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Hamiltonian balance equations

Darryl D. Holm [1]

Theoretical Division and Center for Nonlinear Studies, Los Alamos, National Laboratory, Los Alamos, NM 87545, USA

Abstract

We find Hamiltonian balance equations (HBE) in Eulerian variables in the momentum formulation. This is done by expanding Hamilton's principle (HP) for the primitive equations (PE) in powers of the Rossby number, $\epsilon \ll 1$, truncating at order $O(\epsilon)$, then retaining all the terms that result from taking variations. An alternative set of Hamiltonian balance equations is derived in isopycnal variables (IHBE). The two sets of approximate equations HBE and IHBE differ from PE and isopycnal PE (IPE) at order $O(\epsilon^2)$. Each of these four systems conserves energy and possesses an exact Kelvin theorem, which implies exact potential-vorticity advection. However, HBE and IHBE are balanced, while PE and IPE are not.

Keywords: Balance; Geostrophic flow; Ocean; Atmosphere; Dynamics; Asymptotics

1. Introduction

1.1. Background and problem statement

We say a fluid motion equation is balanced, if specifying the fluid's stratified buoyancy and divergenceless velocity determines its pressure through the solution of an equation which does not contain partial time-derivatives among its highest derivatives. The Euler equations for the incompressible motion of a rotating continuously stratified fluid are balanced in this sense, because the pressure in this case is determined from the buoyancy and velocity of the fluid by the Poisson equation (1.11). However, the hydrostatic approximation of this motion by the primitive equations (PE) is not balanced, because the Poisson equation for the pressure in PE involves the time-derivative of the horizontal velocity divergence, which alters the mathematical character of the Euler system from which PE is derived and may lead to rapid time dependence [9]. Balanced approximations which eliminate this potentially rapid time dependence have been sought and found, usually by using asymptotic expansions of the solutions of the PE in powers of the small Rossby number, $\epsilon \ll 1$, after decomposing the horizontal velocity $\boldsymbol{u}$ into order $O(1)$ rotational and order $O(\epsilon)$ divergent components, as $\boldsymbol{u} = \hat{z} \times \nabla\psi + \epsilon\nabla\chi$, where ψ and χ are the stream function and velocity potential, respectively, for the horizontal motion. (This is just the Helmholtz decomposition with relative weight ϵ.)

[1] E-mail: dholm@lanl.gov.

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Balance equations (BE) are reviewed in the classic paper of McWilliams and Gent [24]. Succeeding investigations have concerned the well-posedness and other features of various BE models describing continuously stratified oceanic and atmospheric motions. For example, consistent initial boundary value problems for BE are examined in [12] and regimes of validity for BE are determined in [13]. Balanced models in isentropic coordinates are discussed in [14]. The development of methods for the numerical solution of BE [28], and the applications of BE to the study of vortex motion on a β-plane [25,26], and to wind-driven ocean circulation [27] have also been discussed. In studies of continuously stratified incompressible fluids, solutions of BE that retain terms of order O(1) and order O(ϵ) in a Rossby number expansion of the PE solutions have been found to compare remarkably well with numerical simulations of the PE [26]. Discussions of the relation between BE and semigeostrophy have also recently appeared [15,32].

One recurring issue in the literature is that, when truncated at order O(ϵ) in the Rossby number expansion, the BE for continuously stratified fluids conserve energy [23], but do not conserve potential vorticity on fluid parcels. Recently, Allen [3] formulated a set of BE for continuously stratified fluids that retains some terms of order O(ϵ^2) and does conserve potential vorticity on fluid parcels. Allen calls these balance equations "BEM equations", because they are based on momentum equations, rather than on the equation for vertical vorticity, as for the standard BE. One advantage of the momentum formulation of BEM over the vorticity formulation of the original BE is that boundary conditions are more naturally imposed on the fluid's velocity than on its vorticity. A disadvantage of the BEM equations formulated in [3] is the absence of any systematic derivation of them from a higher-level fluid theory.

Here we find Hamiltonian balance equations (HBE) in the momentum formulation by using the weighted Helmholtz decomposition for $\boldsymbol{u}$ and expanding Hamilton's principle (HP) for the PE in powers of the Rossby number, $\epsilon \ll 1$. This expansion is truncated at order O(ϵ), then all terms are retained that result from taking variations. As preparation, we discuss an asymptotic expansion of HP for the Euler–Boussinesq (EB) equations which govern rotating stratified incompressible inviscid fluid flow. This asymptotic expansion of HP for the EB equations has two small dimensionless parameters: the aspect ratio of the shallow domain, α, and the Rossby number, ϵ. Setting α equal to zero in this expansion yields HP for PE. Setting ϵ also equal to zero yields HP for equilibrium solutions in both geostrophic and hydrostatic balance. Setting $\alpha = 0$, substituting the ϵ-weighted Helmholtz decomposition for $\boldsymbol{u}$ and truncating the resulting aysmptotic expansion in ϵ of the HP for the EB equations yields HP for a set of nearly geostrophic HBE. The resulting HBE differ slightly from Allen's BEM equations by certain terms of order O(ϵ^2) which are dropped in [3]. However, a redefinition of pressure due to P. Gent shows the HBE and BEM models are equivalent.

The HBE model is

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\mathrm{R}} + \epsilon^2 u_{\mathrm{R}j} \boldsymbol{\nabla} u_{\mathrm{D}}^j + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \boldsymbol{\nabla} p = 0, \qquad \rho + p_z + \epsilon^2 \boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}z} = 0$$
$$\text{with } \frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t \rho + \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho + \epsilon w \rho_z = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} + \epsilon w_z = 0, \tag{1.1}$$

where the horizontal fluid velocity is given by $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon \boldsymbol{u}_{\mathrm{D}} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi + \epsilon \boldsymbol{\nabla}\chi$, subscript z denotes vertical derivative and the rest of the notation is explained in Section 1.2. Dropping all terms of order O(ϵ^2) from either HBE or BEM recovers the BE discussed in [12,13]. As we have discussed, the HBE retain the order O(ϵ^2) terms in the equations that arise from HP for the PE at order O(ϵ). These order O(ϵ^2) terms retained in the HBE provide the conservation laws due to symmetries of HP at the truncation order O(ϵ). Retaining these terms does not improve the order of accuracy of the HBE to order O(ϵ^2) because the HBE do not retain *all* order O(ϵ^2) terms which could result from the PE via the asymptotic expansion in ϵ using the ϵ-weighted Helmholtz decomposition. However, since the resulting HBE differ from EB and PE only at order O(ϵ^2) and do share the same conservation laws and Hamiltonian structure, they may be valid approximations for times longer than the expected order $O(1/\epsilon)$ for BE.

By their construction from a HP which possesses the classic fluid symmetries, the HBE conserve integrated energy and conserve potential vorticity on fluid parcels. Their Hamiltonian structure endows the HBE with the same type of self-consistency that the PE possess (for the same Hamiltonian reason). After all, the conservation laws in both HBE and PE are not accidental. They correspond to symmetries of the Hamiltonian or Lagrangian for the fluid motion under continuous group transformations in accordance with Noether's theorem. In particular, energy is conserved because the Hamiltonian in both theories does not depend on time explicitly, and potential vorticity is conserved on fluid parcels because the corresponding Hamiltonian is invariant under the infinite set of transformations that relabel the fluid parcels without changing the Eulerian velocity and buoyancy. See, e.g., [31] for a review. The vector fields which generate these relabeling transformations turn out to be the steady flows of the HBE and PE models. By definition, these steady flows leave invariant the Eulerian velocity and buoyancy as they move the Lagrangian fluid parcels along the flow lines. Hence, as a direct consequence of their shared Hamiltonian structure, the steady flows of both HBE and PE are relative equilibria. That is, steady HBE and PE flows are critical points of a sum of conserved quantities, including the (constrained) Hamiltonian. This shared critical-point property enables us, for example, to use the Lyapunov method to investigate stability of relative equilibrium solutions of HBE and PE. See [18] for an application of the Lyapunov method in the Hamiltonian framework to the stability of PE relative equilibria. According to the Lyapunov method, convexity of the constrained Hamiltonian at its critical point (the relative equilibrium) is sufficient to provide a norm that limits the departure of the solution from equilibrium under perturbations. See, e.g., Abarbanel et al. [2] for applications of this method to the Euler equations for incompressible fluid dynamics and Holm et al. [20] for other applications to a range of fluid and plasma theories.

Thus, the HBE possess the same Hamiltonian structure as EB and PE, and differ in their Hamiltonian and conservation laws by small terms of order $O(\epsilon^2)$. Moreover, the HBE conservation laws are fundamentally of the same nature as those of the EB equations and the PE from which they descend. These conserved quantities – particularly the quadratic conserved quantities – may eventually be useful measures of the deviations of the HBE solutions from EB and PE solutions under time evolution starting from similar initial conditions.

The plan of the paper is as follows. Section 1 introduces the dimensionless EB equations and defines the notation and scaling regime in which we work. Section 2 derives the dimensionless EB equations from HP and Section 3 discusses their restriction to the PE, upon setting the aspect ratio (α) of the domain to zero in the equations of motion. Section 3 also rederives the PE by making this same restriction in the HP for EB, thereby setting up the theme of Rossby-number expansions in HP by which we derive the HBE in Section 4. Section 4 also discusses the conservation laws, Kelvin circulation theorem and Hamiltonian formulation for the HBE. Section 5 compares the HBE with other BE in the literature and discusses solution procedures for the HBE based on their relation to the BEM equations of Ref. [3]. Section 6 gives explicit expressions for the Kelvin circulation theorem, potential-vorticity advection and energy conservation for HBE. Section 7 presents the Hamiltonian formulation of the HBE in the Eulerian representation and shows that the HBE steady flows are relative equilibria, by showing that they are critical points of the sum of the Hamiltonian and the conserved functionals of buoyancy and potential vorticity for HBE. Section 8 begins our study of isopycnal HBE, by first discussing the isopycnal representation of the PE. Section 9 transforms the HBE into the isopycnal representation and then presents an alternative isopycnal HBE (denoted IHBE) in which the pressure equation attains a much simpler form than for the direct transformation of the HBE from Eulerian to isopycnal coordinates. The IHBE model is derived by making the ϵ-weighted Helmholtz decomposition of the horizontal fluid velocity in terms of gradients along level surfaces of buoyancy ρ (instead of level surfaces of height, h) and using the layer thickness h_ρ as the measure in the inner product in which orthogonality is defined. The IHBE motion equations are given by

$$
\begin{aligned}
&\epsilon \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{u}}_{\mathrm{R}} - \epsilon^2 \tilde{u}_{\mathrm{D}j} \tilde{\boldsymbol{\nabla}} \tilde{u}^j_{\mathrm{R}} + f\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\nabla}}(\tilde{p} + \epsilon^2 \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{D}} + \rho h) = 0, \\
&\rho h_\rho + (\tilde{p} + \epsilon^2 \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{D}})_\rho = 0,
\end{aligned} \tag{1.2}
$$

where tilde $\sim$ denotes dependence on (x, y, ρ, t) and subscript ρ denotes partial derivative. The nondivergent and divergent components of the horizontal fluid velocity in isopycnal coordinates are expressed as

$$
\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_{\mathrm{R}} + \epsilon \tilde{\boldsymbol{u}}_{\mathrm{D}} = \hat{\boldsymbol{z}} \times \tilde{\boldsymbol{\nabla}} \tilde{\psi} + \frac{\epsilon}{h_\rho} \tilde{\boldsymbol{\nabla}} \tilde{\chi}, \tag{1.3}
$$

where $\tilde{\psi}$ and $\tilde{\chi}$ are the stream function and "velocity potential", respectively, for $\tilde{\boldsymbol{u}}$. The IHBE are closed by the kinematic conditions,

$$
\begin{aligned}
&\epsilon \tilde{w} = \frac{\mathrm{d}h}{\mathrm{d}t} = \partial_t h + \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}} h \qquad \text{(the definition of vertical velocity)}, \\
&\partial_t h_\rho = -\tilde{\boldsymbol{\nabla}} \cdot h_\rho \tilde{\boldsymbol{u}} \qquad\qquad\;\; \text{(incompressibility)},
\end{aligned} \tag{1.4}
$$

in which $|\partial_t h|$ and $|\tilde{\boldsymbol{\nabla}} h|$ are order $\mathrm{O}(\epsilon)$.

The IHBE model differs at order $\mathrm{O}(\epsilon^2)$ from the Isopycnal BE of Gent and McWilliams [14] by the effects of the weight $1/h_\rho$ in $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ which lead to an order $\mathrm{O}(\epsilon^2)$ different circulation theorem while producing an exactly conserved energy not present in the Isopycnal BE. See Section 9 for full details. Section 10 discusses the Lie–Poisson Hamiltonian formulation of IHBE which results from its derivation via HP. Finally, Section 11 discusses IHBE in comparison to the isopycnal primitive equations (IPE) and Isopycnal BE.

1.2. Dimensionless Euler–Boussinesq (EB) equations

We consider the motion of a rotating continuously stratified ideal incompressible fluid governed by the adiabatic inviscid Euler equations, in which the effects of buoyancy are treated in the Boussinesq approximation and the Coriolis parameter is allowed to vary spatially. We use dimensionless variables in Cartesian coordinates (x, y, z) to write the Euler equations in the Boussinesq approximation (EB equations) as

$$
\begin{aligned}
&\boldsymbol{\nabla}_3 \cdot \boldsymbol{u}_3 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{u} + \epsilon w_z = 0, \qquad \epsilon \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \boldsymbol{\nabla} p = 0, \\
&\epsilon^2 \alpha^2 \frac{\mathrm{d}w}{\mathrm{d}t} + \rho + p_z = 0, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0.
\end{aligned} \tag{1.5}
$$

The notation used here reflects the asymmetry between horizontal and vertical directions induced by the strong tendency of the solutions of these equations to remain in hydrostatic and geostrophic balance. These balances are enforced physically by gravity and rotation about the vertical and are obtained formally by setting $\alpha = 0$ and $\epsilon = 0$, respectively, in Eq. (1.5) In this notation, three-dimensional vectors and gradient operators have subscript 3, while horizontal vectors and gradient operators are left unadorned. Thus, we denote

$$
\begin{aligned}
&\boldsymbol{x}_3 = (x, y, z), \qquad \boldsymbol{x} = (x, y, 0), \\
&\boldsymbol{u}_3 = (u, v, \epsilon w), \qquad \boldsymbol{u} = (u, v, 0), \\
&\boldsymbol{\nabla}_3 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right), \\
&\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{u}_3 \cdot \boldsymbol{\nabla}_3 = \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} + \epsilon w \partial_z,
\end{aligned} \tag{1.6}
$$

and $\hat{z}$ is the unit vector in the vertical z direction. Without risk of confusion, we denote vertical derivatives such as $w_z \equiv \partial w/\partial z$ and $p_z \equiv \partial p/\partial z$ in Eq. (1.5) with subscript z. Note that the vertical component of the three-dimensional velocity vector $\boldsymbol{u}_3$ in the set (1.6) is weighted by the parameter ϵ (Rossby number).

We form dimensionless variables using the scales (L, B, U_0, f_0) for, respectively, horizontal length, vertical depth, horizontal velocity, and Coriolis parameter. The dimensionless Rossby number ϵ and aspect ratio α, are both expected to be small parameters,

$$\epsilon = \frac{U_0}{f_0 L} \ll 1 \quad \text{and} \quad \alpha = \frac{B}{L} \ll 1. \tag{1.7}$$

Denoting dimensional variables with primes gives the following scale relations:

$$\begin{aligned}
&(x, y) = (x', y')/L, \qquad z = z'/B, \\
&(u, v) = (u', v')/U_0, \qquad \epsilon w = w' L/(U_0 B), \\
&t = t' U_0/L, \qquad f(x, y) = f'(x', y')/f_0, \\
&p = p'/p_C, \quad \text{with } p_C = \rho_0 U_0 f_0 L = \rho_0 U_0^2/\epsilon, \\
&\rho = \rho'/\rho_C, \quad \text{with } \rho_C = p_C/(gB) = \rho_0 F^2/\epsilon,
\end{aligned} \tag{1.8}$$

where ρ_0 is the constant reference density, g is the constant acceleration of gravity and $F = U_0/\sqrt{gB}$ is the Froude number, $F \ll 1$. (In terms of the external Rossby deformation radius, $L_R = \sqrt{gB}/f_0$, the Froude number is also expressed as $F = \epsilon L/L_R$.) Thus, in Eq. (1.5), the quantities $(u, v, \epsilon w)$ are the dimensionless velocity components in the (x, y, z) coordinate directions, t is time, and f is the dimensionless Coriolis parameter. In addition, the quantities p and ρ are the dimensionless pressure and buoyancy, respectively.

We are primarily interested in the case of small Rossby number and shallow fluids, $\epsilon \ll 1, \alpha \ll 1$, with slow spatial variations in dimensionless Coriolis parameter $f(\epsilon \boldsymbol{x})$ and bottom topography $z = -b(\epsilon \boldsymbol{x})$ so that

$$|\nabla f| = O(\epsilon) \quad \text{and} \quad |\nabla b| = O(\epsilon). \tag{1.9}$$

The boundaries at the surface, $z = 0$, the slowly varying bottom, $z = -b(\epsilon \boldsymbol{x})$, and the sidewalls are all taken to be rigid, so the fluid velocity satisfies $\boldsymbol{u}_3 \cdot \hat{\boldsymbol{n}}_3 = 0$ at the boundary with rigid lid, sidewalls and bottom, where $\hat{\boldsymbol{n}}_3$ is the three-dimensional outward unit normal vector at the boundary. For simplicity, the sidewalls of the domain are taken to be *vertical*. Thus, in this geometry, the boundary conditions are expressed as

$$\epsilon w|_{z=0} = 0, \qquad \epsilon w|_{z=-b(\epsilon \boldsymbol{x})} = -\boldsymbol{u} \cdot \nabla b, \qquad \boldsymbol{u} \cdot \hat{\boldsymbol{n}}|_{\text{sides}} = 0, \tag{1.10}$$

where $\hat{\boldsymbol{n}}$ is the horizontal outward unit normal vector at the boundary. The pressure is determined in the EB theory by requiring $\partial_t \nabla_3 \cdot \boldsymbol{u}_3 = 0$ and solving for the p in the dimensionless Poisson equation which results from taking the 3-divergence of the motion equation in the set (1.5). Namely,

$$\Delta p + (p_{zz} + \rho_z)/\alpha^2 + \nabla_3 \cdot \left[\epsilon(\boldsymbol{u}_3 \cdot \nabla_3)\boldsymbol{u}_3 + f\hat{\boldsymbol{z}} \times \boldsymbol{u}\right] = 0, \tag{1.11}$$

where Δ is the Laplacian in the horizontal coordinates. (Note the scaling in z by α and subscript notation for z derivatives.) This equation is to be solved with Neumann boundary conditions which are obtained by evaluating the normal component of the horizontal EB motion equation at each vertical sidewall. Thus, in terms of the normal derivative $p_n = \partial p/\partial n = \hat{\boldsymbol{n}} \cdot \nabla p$, one imposes

$$p_n + \hat{\boldsymbol{n}} \cdot f\hat{\boldsymbol{z}} \times \boldsymbol{u} = \epsilon \boldsymbol{u}_3 \cdot (\nabla_3 \hat{\boldsymbol{n}}_3) \cdot \boldsymbol{u}_3 = \epsilon \boldsymbol{u} \cdot (\nabla \hat{\boldsymbol{n}}) \cdot \boldsymbol{u} = -\epsilon\kappa(\boldsymbol{u} \cdot \hat{\boldsymbol{s}})^2, \tag{1.12}$$

at each lateral boundary, where $\hat{\boldsymbol{s}}$ is the horizontal unit tangent vector (which is unique up to a sign) and κ is the (only) curvature in the horizontal plane. We assume that the unit vertical vector $\hat{\boldsymbol{z}}$, the outward unit normal vector

$\hat{\boldsymbol{n}}$, and the unit tangent vector $\hat{\boldsymbol{s}}$ form a right-handed orthonormal coordinate frame on the boundary, so $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{s}}$. Consequently, we may rewrite the boundary condition (1.12) as

$$p_n - (f - \epsilon\kappa \boldsymbol{u} \cdot \hat{\boldsymbol{s}})\boldsymbol{u} \cdot \hat{\boldsymbol{s}} = 0, \tag{1.13}$$

where the horizontal velocity is tangential on the boundary; so $(\boldsymbol{u} \cdot \hat{\boldsymbol{s}})^2 = |\boldsymbol{u}|^2$ there. Thus, the product $\kappa \boldsymbol{u} \cdot \hat{\boldsymbol{s}}$ of the curvature of the vertical lateral boundary of the domain and the tangential velocity of the fluid appears in the pressure boundary condition as a shift in the Coriolis parameter. (This curvature term in the pressure boundary condition was pointed out to the author by C.D. Levermore.) In the case of the EB equations, solving the Poisson equation (1.11) with Neumann boundary conditions (1.13) determines the pressure p from the buoyancy ρ and the fluid velocity components, $\boldsymbol{u}$ and w. So the EB equations with four degrees of freedom $(\boldsymbol{u}, w, \rho)$ are balanced both in the sense used here and in the sense of removing acoustic waves.

2. HP for ideal fluid dynamics in Eulerian variables

2.1. HP for the dimensionless EB equations

The dimensionless Euler–Boussinesq (EB) equations (1.5) arise from HP, $\delta\mathcal{L}_{\mathrm{EB}} = 0$, under variations of the Lagrangian particle lables $l^A(\boldsymbol{x}_3, t)$, $A = 1, 2, 3$, at constant Eulerian position $\boldsymbol{x}_3$ and time t. The constrained action $\mathcal{L}_{\mathrm{EB}}$ for the HP leading to the EB equations (1.5) is given in dimensional form in [1,19]

$$\mathcal{L}_{\mathrm{EB}} = \int \mathrm{d}t \int \mathrm{d}^3x \big[\tfrac{1}{2}\epsilon\mathcal{D}\big(|\boldsymbol{u}|^2 + \epsilon^2\alpha^2 w^2\big) + \mathcal{D}\boldsymbol{u} \cdot \boldsymbol{R}(\boldsymbol{x}) - \mathcal{D}\rho z - p(\mathcal{D} - 1)\big], \tag{2.1}$$

where $\operatorname{curl} \boldsymbol{R}(\boldsymbol{x}) = f(\epsilon\boldsymbol{x})\hat{\boldsymbol{z}}$, the quantity $\mathcal{D}$ is defined by $\mathcal{D} = \det(\boldsymbol{\nabla}_3 l^A)$, and the pressure p appears as a Lagrange multiplier which enforces incompressibility. By definition, the fluid particle labels $l^A(\boldsymbol{x}_3, t)$, follow the three-dimensional fluid velocity $\boldsymbol{u}_3 = (u, v, \epsilon w)$ and therefore satisfy the advection law,

$$\frac{\mathrm{d}l^A}{\mathrm{d}t} = \frac{\partial l^A}{\partial t} + \boldsymbol{u}_3 \cdot \boldsymbol{\nabla}_3 l^A = \partial_t l^A + \boldsymbol{u} \cdot \boldsymbol{\nabla} l^A + \epsilon w l^A_z = 0, \quad A = 1, 2, 3. \tag{2.2}$$

Hence, we may write the components of the fluid velocity u_3^i, $i = 1, 2, 3$, in terms of partial derivatives of l^A, as

$$u_3^i = -(\mathcal{D}^{-1})^i_A \frac{\partial l^A}{\partial t}, \quad i = 1, 2, 3, \tag{2.3}$$

where we follow the convention of summing on repeated indices over their ranges and $(\mathcal{D}^{-1})^i_A$ is the inverse of $\mathcal{D}^A_i = (\partial l^A / \partial x_3^i)$, the 3×3 Jacobian matrix for the map from Eulerian coordinates to Lagrangian fluid labels. The inverse matrix exists, provided the determinant $\mathcal{D} = \det(\mathcal{D}^A_i)$ does not vanish. This determinant is equal to a constant (taken to be unity) for incompressible flow. Indeed, as a consequence of (2.2), $\mathcal{D}$ satisfies the continuity equation

$$\frac{\partial \mathcal{D}}{\partial t} = -\boldsymbol{\nabla}_3 \cdot \mathcal{D}\boldsymbol{u}_3. \tag{2.4}$$

Thus, if $\mathcal{D}$ is initially equal to unity, it will remain so, according to (2.4), provided $\boldsymbol{\nabla}_3 \cdot \boldsymbol{u}_3 = 0$ at all times, as ensured by the Poisson equation (1.11) for pressure.

Varying the action $\mathcal{L}_{\mathrm{EB}}$ in (2.1) at fixed Eulerian coordinate $\boldsymbol{x}_3$ and time t gives

$$\delta\mathcal{L}_{\rm EB} = \int \mathrm{d}t \int \mathrm{d}^3x \big[(\mathcal{D}\epsilon \boldsymbol{u} + \mathcal{D}\boldsymbol{R}(\boldsymbol{x})) \cdot \delta\boldsymbol{u} + \epsilon^3\alpha^2 w \delta w - \mathcal{D}z\delta\rho + \big(\tfrac{1}{2}\epsilon(|\boldsymbol{u}|^2 + \epsilon^2\alpha^2 w^2) + \boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - \rho z - p\big)\delta\mathcal{D} - (\mathcal{D}-1)\delta p \big]. \tag{2.5}$$

This is expressible in terms of variations δl^A with respect to the Lagrange coordinate l^A, by using the following relations derived from the definitions of $\mathcal{D}$, $\boldsymbol{u}_3$, and ρ,

$$\begin{aligned} \delta\mathcal{D} &= \mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A_{,i}, \\ \delta u^i_3 &= -(\mathcal{D}^{-1})^i_A u^j_3 \delta l^A_{,j} - (\mathcal{D}^{-1})^i_A \delta l^A_{,t}, \quad i,j = 1,2,3, \\ \delta\rho &= \frac{\partial\rho}{\partial l^A}\delta l^A = \rho_{,i}(\mathcal{D}^{-1})^i_A \delta l^A. \end{aligned} \tag{2.6}$$

Remark (on indices). The placement of indices – raised or lowered – is immaterial in Cartesian coordinates. We place them to indicate properties under general coordinate transformations. As a rule then, lowered indices sum with raised ones. Subcript-comma notation denotes partial derivatives. For example, $l^A_{,j} = \partial l^A/\partial x^j$ and $l^A_{,t} = \partial l^A/\partial t$. The comma is dropped for simplicity in denoting partial derivatives of familiar quantities such as p_z and h_ρ where no confusion can arise.

Upon substituting the definitions of $\delta\rho$, δu^i_3 and $\delta\mathcal{D}$ in terms of δl^A into the variational formula (2.5) and integrating by parts using the tangency conditions (1.10) on the velocity at the fixed boundary – that $\boldsymbol{u}_3 \cdot \hat{\boldsymbol{n}}_3 = 0$ – the variation of the action (2.5) becomes

$$\begin{aligned} \delta\mathcal{L}_{\rm EB} = \int \mathrm{d}t \int \mathrm{d}^3x \big\{ & \delta l^A \big[\partial_t \big(\mathcal{D}(\mathcal{D}^{-1})^i_A(\epsilon u_i + R_i) + \epsilon^2\alpha^2\mathcal{D}(\mathcal{D}^{-1})^3_A w\big) \\ & + \partial_j\big(\mathcal{D}u^j_3\big[(\mathcal{D}^{-1})^i_A(\epsilon u_i + R_i) + \epsilon^2\alpha^2(\mathcal{D}^{-1})^3_A w\big]\big)\big] \\ & + \delta l^A \partial_i\big[\mathcal{D}(\mathcal{D}^{-1})^i_A\big(p + z\rho - \boldsymbol{u}\cdot\boldsymbol{R} - \tfrac{1}{2}\epsilon(|\boldsymbol{u}|^2 + \epsilon^2\alpha^2 w^2)\big)\big] \\ & - \mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A z\rho_{,i} - \delta p(\mathcal{D}-1)\big\} \end{aligned} \tag{2.7}$$

Rearrangement of formula (2.7) using the continuity equation (2.4) for $\mathcal{D}$ and the identities

$$\partial_j\mathcal{D} = \mathcal{D}(\mathcal{D}^{-1})^i_A \partial_j \mathcal{D}^A_i, \quad \big(\mathcal{D}(\mathcal{D}^{-1})^i_A\big)_{,i} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{D}^{-1})^i_A = u^i_{3,j}(\mathcal{D}^{-1})^j_A, \tag{2.8}$$

gives the final expression for the variation of the Euler–Boussinesq action,

$$\begin{aligned} \delta\mathcal{L}_{\rm EB} = \int \mathrm{d}t \int \mathrm{d}^3x \Big\{ & \mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A \Big[\epsilon\frac{\mathrm{d}}{\mathrm{d}t}u_i + \delta^3_i\epsilon^2\alpha^2\frac{\mathrm{d}}{\mathrm{d}t}w + u^j(R_{i,j} - R_{j,i}) + \delta^3_i\rho + p_{,i}\Big] \\ & - (\mathcal{D}-1)\delta p\Big\}. \end{aligned} \tag{2.9}$$

where δ^3_i is the Kronecker delta and the reader is reminded that u_i (without subscript 3) is the *horizontal* fluid velocity, so it only has components $i = 1, 2$. Notice that

$$u^j(R_{i,j} - R_{j,i}) = -(\boldsymbol{u} \times \operatorname{curl}\boldsymbol{R})_i = (f\hat{\boldsymbol{z}} \times \boldsymbol{u})_i, \tag{2.10}$$

since $\boldsymbol{R}$ depends only on the horizontal Eulerian position. $\boldsymbol{x}$. Vanishing of $\delta\mathcal{L}_{\mathrm{EB}}$ in (2.9) for arbitrary variations δl^A and δp within the domain of flow now implies the dimensionless EB equations (1.5). This is HP for the EB equations in Eulerian variables.

2.2. *More general forms of the action for Eulerian ideal fluid dynamics*

For any action $\mathcal{L}$ depending on the fluid variables l^A only through the quantities $\boldsymbol{u}_3$, $\mathcal{D}$ and ρ, and with incompressibility imposed by constraining $\mathcal{D}$ to be constant, HP gives

$$\begin{aligned}\delta\mathcal{L} = &\int \mathrm{d}t \int \mathrm{d}^3x \left\{ \mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A \left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta u^i_3} + \frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta u^j_3} u^j_{3,i} - \left(\frac{\delta\mathcal{L}}{\delta\mathcal{D}}\right)_{,i} + \frac{1}{\mathcal{D}}\rho_{,i}\frac{\delta\mathcal{L}}{\delta\rho}\right] - (\mathcal{D}-1)\delta p \right\} \\ &- \int \mathrm{d}t \int \mathrm{d}^3x \left\{ \partial_t \left[\mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A \frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta u^i_3}\right] + \frac{\partial}{\partial x^j_3}\left[\mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A \left(-\frac{\delta\mathcal{L}}{\delta\mathcal{D}}\delta^j_i + \frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta u^i_3} u^j_3\right)\right]\right\}. \end{aligned} \tag{2.11}$$

Vanishing of the coefficient of δl^A gives the motion equation, vanishing of the coefficient of δp gives incompressibility, and vanishing of the exact-derivative terms gives the boundary conditions. From this motion equation, one obtains the Kelvin circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma(t)} \frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta\boldsymbol{u}_3}\cdot \mathrm{d}\boldsymbol{x}_3 + \oint_{\gamma(t)} \frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta\rho}\,\mathrm{d}\rho = 0, \tag{2.12}$$

where the contour $\gamma(t)$ moves with the fluid velocity $\boldsymbol{u}_3$ and $\mathcal{D} = 1$ is imposed by the Lagrange multiplier, p, the pressure. The Kelvin circulation theorem (2.12) is a general result of HP for any action $\mathcal{L}$ expressed in the Eulerian fluid variables $(\boldsymbol{u}_3, \mathcal{D}, \rho)$. Thus, one may develop approximate Eulerian fluid models which possess a Kelvin circulation theorem, simply by making approximations in the original HP. This is our strategy here.

The quantity $\delta\mathcal{L}/\delta\boldsymbol{u}_3$ appearing in the circulation integral in Kelvin's theorem (2.12) plays an important role in the Hamiltonian formulation of ideal fluid dynamics in Eulerian variables. Indeed, the chain rule and relation (2.3) for the velocity $\boldsymbol{u}_3$ imply $\delta\mathcal{L}/\delta\boldsymbol{u}_3 = -\pi_A\boldsymbol{\nabla}_3 l^A$, where $\pi_A = \delta\mathcal{L}/\delta l^A_t$ is the momentum density canonically conjugate to l^A in the Hamiltonian formulation. Consequently, the circulation integral in (2.12) becomes

$$\oint_{\gamma(t)} \frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta\boldsymbol{u}_3}\cdot\mathrm{d}\boldsymbol{x}_3 = -\oint_{\gamma(t)} \frac{\pi_A}{\mathcal{D}}\,\mathrm{d}l^A, \tag{2.13}$$

and $\mathcal{D} = 1$. Hence, Kelvin's theorem (2.12) states that the circulation integral (2.13) is invariant when the fluid contour $\gamma(t)$ lies on a level surface of ρ. In the Hamiltonian formulation, this invariance characterizes the class of fluid flows generated by HP with action $\mathcal{L}(\boldsymbol{u}_3, \mathcal{D}, \rho)$ in the same way that invariance of Poincaré's action integral $\oint p\,\mathrm{d}q$ characterizes the Hamiltonian dynamics of classical particles – as canonical transformations in phase space. Thus, from the Hamiltonian viewpoint, Kelvin's theorem is the geometrical statement of invariance of the fluid action integral $\oint \pi_A\,\mathrm{d}l^A$ on level surfaces of buoyancy. (Incidentally, this geometrical statement takes the same form in Lagrangian fluid variables – invariance of $\oint \boldsymbol{\Pi}\cdot\mathrm{d}\boldsymbol{X}$ on level surfaces of buoyancy, where $\boldsymbol{\Pi}$ is the momentum canonically conjugate to the fluid trajectory $\boldsymbol{X}(l^A, t)$.)

Applying Stokes' theorem to the relation (2.12) on a surface of constant buoyancy $S(t)|_\rho$ with boundary $\gamma(t)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{S(t)|_\rho} \boldsymbol{\nabla}_3 \times \left(\frac{1}{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta\boldsymbol{u}_3}\right)\cdot\boldsymbol{\nabla}_3\rho\frac{\mathrm{d}S}{|\boldsymbol{\nabla}_3\rho|} = 0. \tag{2.14}$$

Thus, the flux of this curl (which turns out to be the absolute vorticity in fluid applications) through a surface of constant buoyancy is invariant under the incompressible flow whose motion equation arises from variations of $l^A(\boldsymbol{x}_3, t)$ in HP, Eq. (2.11). Moreover, Kelvin's theorem and advection of the buoyancy ρ together imply the advection law for potential vorticity,

$$\frac{\partial Q}{\partial t} + \boldsymbol{u}_3 \cdot \nabla_3 Q = 0 \quad \text{with } Q \equiv \frac{1}{\mathcal{D}} \nabla_3 \times \left(\frac{1}{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \boldsymbol{u}_3} \right) \cdot \nabla_3 \rho. \tag{2.15}$$

Thus, advection of potential vorticity is also a general result of HP for any action of the form $\mathcal{L}(\boldsymbol{u}_3, \mathcal{D}, \rho)$. This property may also be obtained directly from the three-dimensional curl of the motion equation arising from HP in (2.11), i.e., from the "vorticity equation",

$$\frac{\partial}{\partial t} \nabla_3 \times \left(\frac{1}{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \boldsymbol{u}_3} \right) - \nabla_3 \times \left(\boldsymbol{u}_3 \times \nabla_3 \times \left(\frac{1}{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \boldsymbol{u}_3} \right) \right) = \nabla_3 \left(\frac{1}{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \rho} \right) \times \nabla_3 \rho. \tag{2.16}$$

Finally, the exact derivative terms in the variational equation (2.11) vanish, provided: (1) the variations δl^A vanish at the endpoints in time; and (2) either $\boldsymbol{u}_3 \cdot \hat{\boldsymbol{n}}_3 = 0 = \delta \boldsymbol{x}_3 \cdot \hat{\boldsymbol{n}}_3$ at the fixed boundaries (as we assume, with $\delta x_3^i \equiv (\mathcal{D}^{-1})^i_A \delta l^A$), or the following condition is satisfied:

$$-\frac{\delta \mathcal{L}}{\delta \mathcal{D}} \hat{\boldsymbol{n}}_3 + \frac{1}{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \boldsymbol{u}_3} (\boldsymbol{u}_3 \cdot \hat{\boldsymbol{n}}_3) = 0, \tag{2.17}$$

at free boundaries (i.e., moving boundaries without traction). This is the starting point for generalizing the present considerations from fixed to moving boundaries in Eulerian variables.

3. Restriction of the EB to the primitive equations (PE)

Setting the aspect ratio parameter α to zero in the EB equations (1.5) yields the dimensionless primitive equations (PE) of ocean and atmosphere models; namely,

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} + \epsilon w_z = 0, \qquad \epsilon \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla p = 0, \\ &\rho + p_z = 0, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0. \end{aligned} \tag{3.1}$$

Setting $\alpha = 0$ in the EB vertical motion equation (1.5) imposes hydrostatic balance in the PE, which allows the pressure p to be determined from the vertical integral of the buoyancy ρ, by the third equation in the set (3.1). The vertical velocity w is determined in the PE from the vertical integral of the continuity equation, the first equation in the set (3.1). Thus, the pressure p and the vertical velocity w in the PE are both diagnostic variables. We shall see later from the viewpoint of HP that p and w are both *Lagrange multipliers*, which impose the constraints of incompressibility and hydrostaticity, respectively. Thus, the PE system, written in terms of the five variables ρ, p, and $\boldsymbol{u}_3$, with two constraints, has only three remaining degrees of freedom, in agreement with the number of partial time derivatives in the PE (3.1).

The PE form a dynamical system which clearly inherits its structure from the EB equations. For example, the Kelvin circulation theorem and the conservation laws for potential vorticity and energy in the EB system lead to those for PE, as follows. Let us define the α^2-weighted three-dimensional velocity $\boldsymbol{v}_3 = (u, v, \epsilon\alpha^2 w)$. Then the EB equations (1.5) are expressed more compactly in the equivalent form

$$\begin{aligned} &\epsilon \partial_t \boldsymbol{v}_3 - \boldsymbol{u}_3 \times \mathrm{curl}_3(\epsilon \boldsymbol{v}_3 + \boldsymbol{R}) + \rho\hat{\boldsymbol{z}} + \nabla_3\left(p + \tfrac{1}{2}\epsilon \boldsymbol{u}_3 \cdot \boldsymbol{v}_3\right) = 0, \\ &\nabla_3 \cdot \boldsymbol{u}_3 = 0 \quad \text{and} \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0. \end{aligned} \tag{3.2}$$

The PE emerge from these equations when $\alpha \to 0$ and, consequently, $\boldsymbol{v}_3 \to \boldsymbol{u}$.

3.1. Kelvin theorem for PE

The EB motion equation in (3.2) implies the following Kelvin circulation theorem for any closed curve $\gamma(t)$ moving with the fluid;

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)}(\epsilon\boldsymbol{v}_3+\boldsymbol{R})\cdot\mathrm{d}\boldsymbol{x}_3 &= \oint_{\gamma(t)}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\boldsymbol{v}_3+\boldsymbol{R})+(\epsilon v_{3j}+R_j)\boldsymbol{\nabla}_3u_3^j\right]\cdot\mathrm{d}\boldsymbol{x}_3\\ &= -\oint_{\gamma(t)}\rho\hat{z}\cdot\mathrm{d}\boldsymbol{x}_3+\oint_{\gamma(t)}\boldsymbol{\nabla}_3\big(-p+\tfrac{1}{2}\epsilon\boldsymbol{u}_3\cdot\boldsymbol{v}_3+\boldsymbol{u}_3\cdot\boldsymbol{R}\big)\cdot\mathrm{d}\boldsymbol{x}_3\\ &= -\oint_{\gamma(t)}\rho\,\mathrm{d}z,\end{aligned} \tag{3.3}$$

where we have re-expressed the EB motion equation using the fundamental vector identity of fluid mechanics,

$$-\boldsymbol{U}\times\mathrm{curl}_3\,\boldsymbol{V}=(\boldsymbol{U}\cdot\boldsymbol{\nabla}_3)\boldsymbol{V}-U^j\boldsymbol{\nabla}_3V_j, \tag{3.4}$$

which holds for any three-component vectors $\boldsymbol{U}$ and $\boldsymbol{V}$. Setting $\alpha = 0$ replaces $\boldsymbol{v}_3$ by $\boldsymbol{u}$ in Eq. (3.3) and thereby produces the Kelvin theorem for the PE.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)}(\epsilon\boldsymbol{u}+\boldsymbol{R})\cdot\mathrm{d}\boldsymbol{x}_3 &= \oint_{\gamma(t)}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\boldsymbol{u}+\boldsymbol{R})+(\epsilon u_j+R_j)\boldsymbol{\nabla}_3u^j\right]\cdot\mathrm{d}\boldsymbol{x}_3\\ &= -\oint_{\gamma(t)}\rho\hat{z}\cdot\mathrm{d}\boldsymbol{x}_3+\oint_{\gamma(t)}\boldsymbol{\nabla}_3\big(-p+\tfrac{1}{2}\epsilon|\boldsymbol{u}|^2+\boldsymbol{u}\cdot\boldsymbol{R}\big)\cdot\mathrm{d}\boldsymbol{x}_3\\ &= -\oint_{\gamma(t)}\rho\,\mathrm{d}z.\end{aligned} \tag{3.5}$$

An application of Stokes' theorem in Eq. (3.3) implies that the flux of total vorticity $\mathrm{curl}_3(\epsilon\boldsymbol{v}_3+\boldsymbol{R})$ through a surface of constant ρ is invariant. From the viewpoint of the Hamiltonian formulation, the Kelvin circulation property of the EB and PE systems is a fundamental property of Eulerian ideal fluid dynamics. Indeed, any approximate ideal fluid model misses being a fluid theory to the extent that it sacrifices the Kelvin circulation property. Fortunately, as we have seen in Section 2.2, this property is preserved by any approximate ideal fluid model derived from HP with action $\mathcal{L}(\boldsymbol{u}_3, \mathcal{D}, \rho)$. Thus, HP asymptotics for such an action is guaranteed to produce an ideal fluid model which possesses the Kelvin circulation property.

3.2. Potential vorticity for PE

The three-dimensional curl of the EB motion equation in (3.2) yields the total vorticity equation

$$\frac{\partial}{\partial t}\mathrm{curl}_3(\epsilon\boldsymbol{v}_3+\boldsymbol{R})-\mathrm{curl}_3\,(\boldsymbol{u}_3\times\mathrm{curl}_3(\epsilon\boldsymbol{v}_3+\boldsymbol{R}))=\hat{z}\times\boldsymbol{\nabla}_3\rho. \tag{3.6}$$

Combining this result with the remaining two equations in (3.2) gives

$$\frac{\mathrm{d}Q_{\mathrm{EB}}}{\mathrm{d}t}=0\quad\text{with }Q_{\mathrm{EB}}\equiv\mathrm{curl}_3(\epsilon\boldsymbol{v}_3+\boldsymbol{R})\cdot\boldsymbol{\nabla}_3\rho. \tag{3.7}$$

When we restrict to $\alpha = 0$, we replace $\boldsymbol{v}_3$ with $\boldsymbol{u} = (u, v, 0)$ and (3.7) becomes the advection law for the PE potential vorticity, given by

$$\frac{\mathrm{d}Q_{\mathrm{PE}}}{\mathrm{d}t} = 0 \quad \text{with } Q_{\mathrm{PE}} \equiv \mathrm{curl}_3(\epsilon\boldsymbol{u} + \boldsymbol{R}) \cdot \boldsymbol{\nabla}_3\rho. \tag{3.8}$$

Advection of both Q and ρ (where Q is either Q_{EB}, or Q_{PE}), combined with the weighted divergence condition and tangential boundary conditions on $\boldsymbol{u}_3$, yields an infinity of conserved quantities

$$C_\Phi = \int \mathrm{d}^3x\, \Phi(Q, \rho), \tag{3.9}$$

for any function Φ, and for either the EB, or the PE fluid theories.

3.3. Energy conservation for PE

Taking the dot product of the EB motion equation in (3.2) with the velocity $\boldsymbol{u}_3 = (u, v, \epsilon w)$ and summing with z times the buoyancy equation gives energy conservation for the EB equations, in the form

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\epsilon\boldsymbol{u}_3 \cdot \boldsymbol{v}_3 + \rho z\right) + \boldsymbol{\nabla}_3 \cdot \left(p + \tfrac{1}{2}\epsilon\boldsymbol{u}_3 \cdot \boldsymbol{v}_3 + \rho z\right)\boldsymbol{u}_3 = 0. \tag{3.10}$$

In the case $\alpha = 0$, this EB energy conservation equation restricts to that for the PE, again by replacing $\boldsymbol{v}_3$ with $\boldsymbol{u}$. Namely,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\epsilon|\boldsymbol{u}|^2 + \rho z\right) + \boldsymbol{\nabla}_3 \cdot \left(p + \tfrac{1}{2}\epsilon|\boldsymbol{u}|^2 + \rho z\right)\boldsymbol{u}_3 = 0, \tag{3.11}$$

where p is now the pressure for the PE. The energy integral,

$$E = \int \mathrm{d}^3x\left(\tfrac{1}{2}\epsilon|\boldsymbol{u}|^2 + z\rho\right), \tag{3.12}$$

is then time independent for PE solutions satisfying the boundary conditions (1.10).

The Kelvin circulation theorem (3.5) and the conserved quantities E and C_Φ for the PE are evidently the $\alpha = 0$ legacy of the corresponding properties for the EB. We shall see later (in Section 7) how to interpret E and C_Φ from the viewpoint of the Hamiltonian formalism.

3.4. HP for the dimensionless PE

The dimensionless constrained action $\mathcal{L}_{\mathrm{EB}}$ in (2.1) which leads via HP to the EB equations (1.5) is expressed in terms of the α^2-weighted velocity $\boldsymbol{v}_3 = (u, v, \epsilon\alpha^2 w)$ as

$$\mathcal{L}_{\mathrm{EB}} = \int \mathrm{d}t \int \mathrm{d}^3x\left[\tfrac{1}{2}\epsilon\mathcal{D}\boldsymbol{u}_3 \cdot \boldsymbol{v}_3 + \mathcal{D}\boldsymbol{u} \cdot \boldsymbol{R}(\boldsymbol{x}) - \mathcal{D}\rho z - p(\mathcal{D} - 1)\right], \tag{3.13}$$

where $\boldsymbol{u}_3 = (u, v, \epsilon w)$. Setting the aspect ratio parameter α to zero in $\mathcal{L}_{\mathrm{EB}}$ in (3.13) immediately provides the action in HP for the dimensionless PE (3.1),

$$\mathcal{L}_{\mathrm{PE}} = \int \mathrm{d}t \int \mathrm{d}^3x\left[\tfrac{1}{2}\epsilon\mathcal{D}|\boldsymbol{u}|^2 + \mathcal{D}\boldsymbol{u} \cdot \boldsymbol{R}(\boldsymbol{x}) - \mathcal{D}\rho z - p(\mathcal{D} - 1)\right]. \tag{3.14}$$

Varying $\mathcal{L}_{\mathrm{PE}}$ with respect to the fluid particle labels $l^A(\boldsymbol{x}_3, t)$, with $A = 1, 2, 3$, at fixed Eulerian position $\boldsymbol{x}_3$ and time t now leads to the PE, upon retracing the steps taken between Eq. (2.5) and (2.10) in deriving the EB equations.

Thus, from the viewpoint of HP, imposition of hydrostatic balance corresponds to ignoring the kinetic energy of vertical motion by setting $\alpha = 0$ in the EB action (2.1).

Remark. In the other limit, $\epsilon \to 0$, HP for $\mathcal{L}_{\mathrm{EB}}$ (cf. Eq. (2.9)) gives

$$f\hat{z} \times \boldsymbol{u} + \hat{z}\rho + \boldsymbol{\nabla}_3 p = 0, \tag{3.15}$$

which enforces both hydrostatic and geostrophic equilibrium balance.

4. Balance equations via HP

4.1. Setup

Balance equations (BE) approximate the PE by using asymptotic expansions and an ϵ-weighted Helmholtz decomposition of the horizontal fluid velocity $\boldsymbol{u} = \hat{z} \times \boldsymbol{\nabla}\psi + \epsilon\boldsymbol{\nabla}\chi$ to remove the potentially high-frequency compression waves which arise in deriving PE, as $\alpha \to 0$ while imposing the hydrostatic approximation on the EB equations. The source of these waves is the order $\mathrm{O}(\epsilon^2)$ partial time-derivative of the divergence of the horizontal velocity in the Poisson equation for the pressure,

$$-\Delta p = \epsilon\partial_t \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{\nabla} \cdot \left[\epsilon(\boldsymbol{u}_3 \cdot \boldsymbol{\nabla}_3)\boldsymbol{u} + f\hat{z} \times \boldsymbol{u}\right] \quad \text{with } \boldsymbol{\nabla} \cdot \boldsymbol{u} = -\epsilon w_z, \tag{4.1}$$

obtained from the divergence of the PE motion equation in (3.1). Removing this term to higher order in ϵ establishes a balanced pressure equation, which determines the pressure from the velocity without using time derivatives. Apparently, the accuracy of BE compared to the PE depends on the Rossby number being small, $\epsilon \ll 1$. Thus, we are in the realm of approximation theory and in possession of the small dimensionless parameter ϵ. In making approximations, though, we wish to preserve two potentially opposing structural aspects of the original EB equations: their balanced nature and their Hamiltonian property. Our approach in doing this is to continue the theme of small-Rossby-number expansions in HP discussed in Section 3. The approximate equations we derive this way will have the advantage of being Hamiltonian systems. Such systems will conserve energy and potential vorticity at each order in the ϵ-expansion, because they preserve the fundamental symmetry properties of HP for fluids. They will also admit Hamiltonian methods for their subsequent analysis. Thus, our strategy is to first seek approximations of the PE that are Hamiltonian, then to check whether they are also balanced. A similar strategy is used in [30] for deriving approximate models of rotating shallow water dynamics which satisfy energy and potential-vorticity conservation.

BE modeling begins, [12–14], by expressing the dimensionless horizontal velocity vector $\boldsymbol{u}$ as the ϵ-weighted sum of its rotational and divergent parts, $\boldsymbol{u}_{\mathrm{R}}$ and $\boldsymbol{u}_{\mathrm{D}}$, respectively,

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon\boldsymbol{u}_{\mathrm{D}} \equiv \hat{z} \times \boldsymbol{\nabla}\psi + \epsilon\boldsymbol{\nabla}\chi, \\ &\quad \text{or} \quad \boldsymbol{u}_{\mathrm{R}} = (u_{\mathrm{R}}, v_{\mathrm{R}}, 0) = (-\psi_y, \psi_x, 0) \;\; \boldsymbol{u}_{\mathrm{D}} = (u_{\mathrm{D}}, v_{\mathrm{D}}, 0) = (\chi_x, \chi_y, 0). \end{aligned} \tag{4.2}$$

Thus, in BE modeling the rotational part of the horizontal velocity is assumed to dominate its divergent part, in proportion to the ratio of the Coriolis force to the nonlinearity in the PE. In the Helmholtz decomposition of $\boldsymbol{u}$, the BE ordering assumption implies that

$$\begin{aligned} &\mathrm{curl}_3\, \boldsymbol{u}_3 = (\epsilon w_y - v_{\mathrm{R}z}, -\epsilon w_x + u_{\mathrm{R}z}, v_{\mathrm{R}x} - u_{\mathrm{R}y}), \\ &\boldsymbol{\nabla}_3 \cdot \boldsymbol{u}_3 = \epsilon(u_{\mathrm{D}x} + u_{\mathrm{D}y}) + \epsilon w_z = 0, \end{aligned} \tag{4.3}$$

and, thus, in terms of ψ and χ,

$$\begin{aligned} \zeta &\equiv \hat{z} \cdot \mathrm{curl}_3\, \boldsymbol{u}_3 = v_{\mathrm{R}x} - u_{\mathrm{R}y} = \psi_{xx} + \psi_{yy} \equiv \Delta\psi, \\ \mathrm{div}\, \boldsymbol{u} &= u_x + v_y = \epsilon(u_{\mathrm{D}x} + v_{\mathrm{D}y}) = \epsilon\Delta\chi. \end{aligned} \tag{4.4}$$

The continuity equation in (4.3) then gives

$$w_z = -\Delta\chi, \tag{4.5}$$

so w is determined from χ and the top boundary condition, $w|_{z=0} = 0$. Vice versa, $\boldsymbol{u}_{\mathrm{D}}$ is determined from w by

$$\boldsymbol{u}_{\mathrm{D}} = -\Delta^{-1}\nabla w_z. \tag{4.6}$$

Hence, variations in $\boldsymbol{u}_{\mathrm{D}}$ are related to variations in w and χ by

$$\delta\boldsymbol{u}_{\mathrm{D}} = -\Delta^{-1}\nabla\partial_z\delta w = \nabla\delta\chi. \tag{4.7}$$

Finding χ from w via the Poisson equation (4.5) requires a boundary condition on χ, in order to invert the horizontal Laplacian in (4.7). The condition that the velocity $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon\boldsymbol{u}_{\mathrm{D}}$ be tangential at the boundary gives $\hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{R}} + \epsilon\hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{D}} = 0$, so one could take $\hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{R}} = \mathrm{O}(\epsilon)$ on the boundary and expand $\boldsymbol{u}_{\mathrm{R}}$ in powers of ϵ to match order $\mathrm{O}(\epsilon)$ terms at the boundary. However, for simplicity, we follow Ref. [12] in setting the normal components of both velocities to zero separately at the boundary. This choice gives the homogeneous Neumann boundary condition, $\hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{D}} = 0$, i.e.,

$$\chi_n = 0 \quad \text{with } \chi_n = \hat{\boldsymbol{n}} \cdot \nabla\chi = \hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{D}}, \tag{4.8}$$

and enforces

$$-\psi_s = \hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{R}} = 0, \tag{4.9}$$

at each (vertical) lateral boundary, where $\hat{\boldsymbol{n}}$ is the two-dimensional outward unit normal vector at each level surface of the height z. This choice is restrictive, since it eliminates Kelvin waves, which may be important in maintaining balance when an interior flow is strongly influenced by waves on the boundaries [12]. Nonetheless, we shall use condition (4.8) in our discussion of the ϵ-expansion of HP in Section 4.2 and decline to discuss the question of more general boundary conditions at this point. Gent and McWilliams [12] offer several alternative choices of boundary conditions for balanced models in a finite domain. Such boundary conditions could be incorporated into the HP approach by using the standard technique of adding a null Lagrangian to L. (A null Lagrangian is the space and time integral of a total divergence, whose only contribution in HP appears at the boundary. See, e.g., [11] for discussion.) However, we do not pursue this approach here.

The partial time-derivative of the boundary condition (4.9) must also vanish, so that

$$\hat{\boldsymbol{n}} \cdot \partial_t\boldsymbol{u}_{\mathrm{R}} = -\psi_{st} = 0 \quad \text{at the boundary}. \tag{4.10}$$

This relation will be useful later, in proving energy conservation for HBE in Section 6.

Counting degrees of freedom. The Poisson equation $-\Delta\chi = w_z$, with $\chi_n = 0$ at the domain boundary, determines χ from w and, thus, the number of degrees of freedom reduces from three for the PE (ψ, ρ, χ) to only two for the HBE (ψ, ρ). The HBE reduction is thus intermediate between the PE and model equations in which the potential vorticity alone is used to determine the velocity, see, e.g., [3,32] for more discussion of the counting of degrees of freedom involved in balance models.

4.2. The ϵ-expansion of HP for EB and PE

The relative ordering in ϵ assumed for $\boldsymbol{u}$ in (4.2) allows us to expand the kinetic energy of horizontal motion $|\boldsymbol{u}|^2 = |\boldsymbol{u}_{\mathrm{R}} + \epsilon \boldsymbol{u}_{\mathrm{D}}|^2$ which appears in the action for the EB equations (2.1) and rewrite this action as a *nested* sum, ordered in powers of ϵ,

$$\begin{aligned}
\mathcal{L}_{\mathrm{EB}} &= \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \epsilon^3\mathcal{L}_3 + \epsilon^3\alpha^2\mathcal{L}_4 \\
&= \int \mathrm{d}t \int \mathrm{d}^3x\, [\mathcal{D}\boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - \mathcal{D}\rho z - p(\mathcal{D}-1)] + \epsilon \int \mathrm{d}t \int \mathrm{d}^3x\, \tfrac{1}{2}\mathcal{D}|\boldsymbol{u} - \epsilon\boldsymbol{u}_{\mathrm{D}}|^2 \\
&\quad + \epsilon^2 \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{D}(\boldsymbol{u} - \epsilon\boldsymbol{u}_{\mathrm{D}})\cdot\boldsymbol{u}_{\mathrm{D}} + \epsilon^3 \int \mathrm{d}t \int \mathrm{d}^3x\, \tfrac{1}{2}\mathcal{D}|\boldsymbol{u}_{\mathrm{D}}|^2 + \epsilon^3\alpha^2 \int \mathrm{d}t \int \mathrm{d}^3x\, \tfrac{1}{2}\mathcal{D}w^2. \qquad (4.11)
\end{aligned}$$

The Rossby-number expansion of HP $\delta\mathcal{L}_{\mathrm{EB}} = 0$ now gives a hierarchy of equations which is ordered in powers of ϵ. Each equation in the hierarchy will be Hamiltonian and will conserve energy and potential vorticity, although not all of them will necessarily be balanced in the sense that we use here that the pressure is solved as a diagnostic variable from an equation with no partial time derivatives appearing at highest derivative order.

4.2.1. $\mathrm{O}(\epsilon)$ variation and the HBE

As in Eq. (3.15), variation of the leading order action gives

$$\delta\mathcal{L}_0 = \int \mathrm{d}t \int \mathrm{d}^3x \big[\mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A (f\hat{\boldsymbol{z}}\times\boldsymbol{u} + \hat{\boldsymbol{z}}\rho + \boldsymbol{\nabla}_3 p)_i - (\mathcal{D}-1)\delta p\big], \qquad (4.12)$$

whose vanishing implies the leading order conditions of geostrophy, hydrostaticity, and incompressibility.

Upon using definitions (2.6) and $\delta\boldsymbol{u}_{\mathrm{D}} = \boldsymbol{\nabla}\delta\chi$ from (4.7), the variation of the next entry in the ϵ-series for the action in (4.11) is expressed as

$$\begin{aligned}
\delta\mathcal{L}_1 &= \int \mathrm{d}t \int \mathrm{d}^3x \big[\tfrac{1}{2}|\boldsymbol{u}_{\mathrm{R}}|^2\delta\mathcal{D} + \mathcal{D}\boldsymbol{u}_{\mathrm{R}}\cdot(\delta\boldsymbol{u} - \epsilon\delta\boldsymbol{u}_{\mathrm{D}})\big] \\
&= \int \mathrm{d}t \int \mathrm{d}^3x \left[\mathcal{D}(\mathcal{D}^{-1})^i_A\delta l^A \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\mathrm{R}} + \epsilon u_{\mathrm{R}j}\boldsymbol{\nabla}_3 u^j_{\mathrm{D}}\right)_i - \epsilon\mathcal{D}\boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{\nabla}\delta\chi\right]. \qquad (4.13)
\end{aligned}$$

The final term in (4.13) may be shown to vanish, by using the divergence theorem. Namely,

$$\begin{aligned}
\int \mathrm{d}t \int \mathrm{d}^3x\mathcal{D}\boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{\nabla}\delta\chi &= \int \mathrm{d}t \int \mathrm{d}^3x\, [\boldsymbol{\nabla}\cdot(\mathcal{D}\boldsymbol{u}_{\mathrm{R}}\delta\chi) - \delta\chi\boldsymbol{\nabla}\cdot(\mathcal{D}\boldsymbol{u}_{\mathrm{R}})], \\
&= \int \mathrm{d}t \int \mathrm{d}z \oint \mathrm{d}s\, \mathcal{D}\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}\delta\chi - \int \mathrm{d}t \int \mathrm{d}^3x\, \delta\chi\boldsymbol{\nabla}\cdot(\mathcal{D}\boldsymbol{u}_{\mathrm{R}}). \qquad (4.14)
\end{aligned}$$

Both terms in (4.14) vanish, one because of the boundary condition $\boldsymbol{u}_{\mathrm{R}}\cdot\hat{\boldsymbol{n}} = 0$ and the other because the expression $\mathcal{D}\boldsymbol{u}_{\mathrm{R}} = \mathcal{D}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi$ has no horizontal divergence when $\mathcal{D} = 1$. Hence, the sum of Eqs. (4.12) and (4.13) gives [cf. Eq. (2.11)]

$$\begin{aligned}
\delta(\mathcal{L}_0 + \mathcal{L}_1) = \int \mathrm{d}t \int \mathrm{d}^3x \bigg\{&\mathcal{D}(\mathcal{D}^{-1})^i_A\delta l^A \left[(f\hat{\boldsymbol{z}}\times\boldsymbol{u} + \hat{\boldsymbol{z}}\rho + \boldsymbol{\nabla}_3 p) + \epsilon\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\mathrm{R}} + \epsilon u_{\mathrm{R}j}\boldsymbol{\nabla}_3 u^j_{\mathrm{D}}\right)\right]_i \\
&-(\mathcal{D}-1)\delta p - \epsilon^2\delta\chi\boldsymbol{\nabla}\cdot\mathcal{D}\boldsymbol{u}_{\mathrm{R}}\bigg\} + \epsilon^2 \int \mathrm{d}t \int \mathrm{d}z \oint \mathrm{d}s\, \mathcal{D}\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}\delta\chi. \qquad (4.15)
\end{aligned}$$

HP now implies the HBE; namely, the $\mathrm{O}(\epsilon^2)$ motion equation in the Rossby-number hierarchy which arises from vanishing of the sum $\delta(\mathcal{L}_0 + \epsilon\mathcal{L}_1)$ in Eq. (4.15). The HBE motion equation is found to be

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\mathrm{R}}+\epsilon^2 u_{\mathrm{R}j}\boldsymbol{\nabla}_3 u_{\mathrm{D}}^j+f\hat{\boldsymbol{z}}\times\boldsymbol{u}+\hat{\boldsymbol{z}}\rho+\boldsymbol{\nabla}_3 p=0. \tag{4.16}$$

In addition, we have

$$\mathcal{D}=1,\qquad \operatorname{div}\mathcal{D}\boldsymbol{u}_{\mathrm{R}}=0,\qquad \hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}=0\quad\text{on the boundary}. \tag{4.17}$$

The kinematic relations that complete the system are

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=0\quad\text{and}\quad \partial_t\mathcal{D}=-\operatorname{div}_3\mathcal{D}\boldsymbol{u}_3. \tag{4.18}$$

These relations follow from advection (2.2) of the Lagrange coordinates and the definitions

$$\rho=\rho(l^A)\quad\text{and}\quad \mathcal{D}=\det(\boldsymbol{\nabla}_3 l^A). \tag{4.19}$$

As an immediate consequence of Eq. (4.18), the constraint $\mathcal{D}=1$ implies $\boldsymbol{\nabla}_3\cdot\boldsymbol{u}_3=0$, so the flow is incompressible. With $\partial\boldsymbol{u}_{\mathrm{D}}/\partial t$ absent in Eq. (4.16), waves due to horizontal compression are absent. Thus, the motion equation in (4.16) is balanced, in the sense that no time-derivatives appear in the equations which determine the pressure, p,

$$\begin{aligned}&\Delta p+\boldsymbol{\nabla}\cdot\left[f\hat{\boldsymbol{z}}\times\boldsymbol{u}+\epsilon(\boldsymbol{u}_3\cdot\boldsymbol{\nabla}_3)\boldsymbol{u}_{\mathrm{R}}+\epsilon^2\big(u_{\mathrm{R}j}\boldsymbol{\nabla}u_{\mathrm{D}}^j\big)\right]=0,\\ &p_z+\rho+\epsilon^2 u_{\mathrm{R}j}u_{\mathrm{D}z}^j=0,\end{aligned} \tag{4.20}$$

where $\Delta=\partial_{xx}+\partial_{yy}$ is the two-dimensional Laplacian. Given ρ, $\boldsymbol{u}_{\mathrm{R}}$ and $\boldsymbol{u}_{\mathrm{D}}$, the Poisson equation in (4.20) for the pressure p is solved with Neumann boundary conditions,

$$\frac{\partial p}{\partial n}+\epsilon^2 u_{\mathrm{R}j}\frac{\partial}{\partial n}u_{\mathrm{D}}^j+\hat{\boldsymbol{n}}\cdot[f\hat{\boldsymbol{z}}\times\boldsymbol{u}+\epsilon(\boldsymbol{u}_3\cdot\boldsymbol{\nabla}_3)\boldsymbol{u}_{\mathrm{R}}]=0, \tag{4.21}$$

applied at each vertical lateral boundary, whose outward unit normal vector $\hat{\boldsymbol{n}}$ is horizontal. By $\boldsymbol{\nabla}_3\cdot\boldsymbol{u}_3=0$ and $\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}=0=\hat{\boldsymbol{n}}\cdot\boldsymbol{u}$ on the boundary, we find

$$\hat{\boldsymbol{n}}\cdot(\boldsymbol{u}_3\cdot\boldsymbol{\nabla}_3)\boldsymbol{u}_{\mathrm{R}}=-u_3^j u_{\mathrm{R}}^i\partial_{3j}\hat{n}_i=-u_{\mathrm{R}}^i\hat{n}_{i,j}u^j=\kappa(\hat{\boldsymbol{s}}\cdot\boldsymbol{u}_{\mathrm{R}})(\hat{\boldsymbol{s}}\cdot\boldsymbol{u}), \tag{4.22}$$

where $\hat{\boldsymbol{s}}$ is the horizontal unit tangent vector and κ is the curvature of the boundary in the horizontal plane. Since $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{n}}$, and $\hat{\boldsymbol{s}}$ are assumed to form a right-handed orthogonal coordinate frame on the boundary, we may then rewrite (4.21) as, cf. Eq. (1.13),

$$p_n+\epsilon^2 u_{\mathrm{R}j}u_{\mathrm{D}n}^j-(f-\epsilon\kappa\psi_n)(\psi_n+\epsilon\chi_s)=0. \tag{4.23}$$

Solving the Poisson equation (4.20) determines the pressure p from the buoyancy ρ and the velocities $\boldsymbol{u}_{\mathrm{R}}$ and $\boldsymbol{u}_{\mathrm{D}}$ for HBE, just as in the case of the EB equations, without the appearance of any time-derivative terms. Thus, the motion equation obtained from the first two terms in the HP ϵ-series is balanced. It turns out that the next-order set of equations obtained from the HP hierarchy is *not* balanced in this way.

Remark (on boundary conditions, constraints, and the weighted Helmholtz decomposition for HBE). The boundary conditions summoned by HP for the HBE (4.16) are expressed in terms of the Helmholtz velocity decomposition (4.2), as

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}=0=\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{D}}\quad\text{at the boundary}, \tag{4.24}$$

and are expressed in terms of the velocity potential χ and stream function ψ, as

$$\psi_s=0=\chi_n\quad\text{at the boundary}. \tag{4.25}$$

In the variation $\delta\mathcal{L}_1$ in Eq. (4.13), the horizontal velocity potential χ looks, at first, like a Lagrange multiplier which could enforce $\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}} = 0$ (or, equivalently, $\psi_s = 0$) on the boundary and $\nabla\cdot(D\boldsymbol{u}_{\mathrm{R}}) = 0$ in the interior region of the flow. However, the variation $\delta\chi$ is related to the vertical velocity variation δw by (4.7) and this, in turn, is related to variations of the Lagrangian coordinates by the formula (2.6) with $i = 3$. After a computation, this last relation implies a *nonlocal* contribution to the equation for $\boldsymbol{u}_{\mathrm{R}}$, which however *vanishes* when the $\boldsymbol{u}_{\mathrm{R}}$-divergence condition $\nabla\cdot(D\boldsymbol{u}_{\mathrm{R}}) = 0$ is satisfied, as imposed by the Helmholtz decomposition (4.2) and the p-constraint, $\mathcal{D} = 1$. At first sight, this convenient simplification appears to be a coincidence. However, recall that the preservation in time of the $\boldsymbol{u}_{\mathrm{R}}$-divergence condition by the horizontal motion equation (in combination with hydrostaticity) determines the pressure, p. Thus, the Helmholtz decomposition (4.2) makes sense as a dynamical constraint. In fact, the functionally related pair, w and χ, may be regarded as a *single* Lagrange multiplier which imposes the $\boldsymbol{u}_R$-divergence condition, $\nabla\cdot(D\boldsymbol{u}_{\mathrm{R}}) = 0$, as a constraint and whose dynamical preservation determines p, the (hydrostatic) pressure. It remains then, to determine w and χ together, by the method of Lagrange multipliers.

We remark also that the $\boldsymbol{u}_{\mathrm{R}}$-divergence condition is consistent with a *weighted* Helmholtz decomposition,

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon\boldsymbol{u}_{\mathrm{D}} = -\nabla\times\psi\hat{\boldsymbol{z}} + \frac{\epsilon}{\mathcal{D}}\nabla\chi, \tag{4.26}$$

according to which $\boldsymbol{u}_{\mathrm{R}}$ and $\boldsymbol{u}_{\mathrm{D}}$ are orthogonal with respect to the measure $\mathcal{D}$, i.e.,

$$\int \mathrm{d}^3x\,\mathcal{D}\boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}} = 0. \tag{4.27}$$

In the present case, the additional constraint $\mathcal{D} = 1$ is imposed by the other Lagrange multiplier, p, the pressure in the action (4.11). So the weight in the Helmholtz decomposition (4.26) turns out to be unity, as appears in Eq. (4.2). Thus, HP provides a self-consistent framework with an orthogonal decomposition of the variables in the Rossby-number expansion of the horizontal velocity. It also gives a solution procedure for these variables (the method of Lagrange multipliers) and natural boundary conditions.

Remark (on the case of a single layer at constant ρ). In the case of a single layer of fluid undergoing columnar motion (i.e., $\partial_z\boldsymbol{u} = 0$) at constant mass density, the $\mathrm{O}(\epsilon)$ action $\mathcal{L}_{\mathrm{EB}}$ in (4.11) reduces to

$$\begin{aligned}\mathcal{L}_{01} &= [\mathcal{L}_0 + \epsilon\mathcal{L}_1]_{1\text{-layer}}\\ &= \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\big[\mathcal{D}\boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - p(\mathcal{D} - b(\epsilon\boldsymbol{x})) + \tfrac{1}{2}\epsilon\mathcal{D}|\boldsymbol{u} - \epsilon\boldsymbol{u}_{\mathrm{D}}|^2\big],\end{aligned} \tag{4.28}$$

where $z = -b(\epsilon\boldsymbol{x})$ at the bottom. The assumption $\partial_z\boldsymbol{u} = 0$ allows us to perform the z-integration in the action (4.11). In this situation, incompressibility, $\nabla_3\cdot\boldsymbol{u}_3 = 0$, and the top boundary condition $w|_{z=0} = 0$, imply that w may be expressed in terms of $\boldsymbol{u}$ as

$$\epsilon w = -z\nabla\cdot\boldsymbol{u}. \tag{4.29}$$

The bottom boundary condition, $\epsilon w|_{z=-b} = -\boldsymbol{u}\cdot\nabla b$, then gives weighted incompressibility

$$\nabla\cdot b\boldsymbol{u} = 0. \tag{4.30}$$

Variations of the single-layer action in (4.28) give

$$\begin{aligned}\delta\mathcal{L}_{01} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\bigg\{&\mathcal{D}(\mathcal{D}^{-1})^i_a\delta l^a\left[\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\mathrm{R}} + \epsilon^2 u_{\mathrm{R}j}\nabla u^j_{\mathrm{D}} + f\hat{\boldsymbol{z}}\times\boldsymbol{u} + \nabla p\right]_i\\ &- (\mathcal{D} - b(\epsilon\boldsymbol{x}))\delta p + \epsilon^2\delta\chi\nabla\cdot b(\epsilon\boldsymbol{x})\boldsymbol{u}_{\mathrm{R}}\bigg\} - \epsilon^2\int \mathrm{d}t\oint \mathrm{d}s\, b(\epsilon\boldsymbol{x})\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}\delta\chi,\end{aligned} \tag{4.31}$$

where $i, a = 1, 2$, and we have integrated by parts as in Eq. (4.15). HP now implies the single-layer equation (cf. the HBE (4.16))

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u}_{\mathrm{R}} + \epsilon^2 u_{\mathrm{R}j} \nabla u_{\mathrm{D}}^j + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla p = 0, \tag{4.32}$$

as well as (cf. Eq. (4.17))

$$\mathcal{D} = b(\epsilon \boldsymbol{x}), \qquad \nabla \cdot b\boldsymbol{u}_{\mathrm{R}} = 0, \qquad \hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{R}} = 0 \quad \text{on the boundary}. \tag{4.33}$$

Finally, we have the kinematic relation

$$\partial_t \mathcal{D} = -\nabla \cdot \mathcal{D}\boldsymbol{u}, \tag{4.34}$$

which completes the system. Inserting the constraint $\mathcal{D} = b(\epsilon \boldsymbol{x})$ with $\partial_t b = 0$ into the last equation recovers the weighted incompressibility condition (4.30) for $\boldsymbol{u}$. Then, since $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon \boldsymbol{u}_{\mathrm{D}}$ and $\boldsymbol{u}_{\mathrm{D}} = \nabla \chi$, the condition $\nabla \cdot b\boldsymbol{u}_{\mathrm{R}} = 0$ in (4.33) which leads to a balanced pressure equation, combined with weighted incompressibility (4.30), implies

$$\nabla \cdot b \nabla \chi = 0; \tag{4.35}$$

whose only solution for $b > 0$ and $\chi_n = 0$ on the boundary is $\chi =$ constant. Hence, $\boldsymbol{u}_{\mathrm{D}} = 0$ and the single-layer HBE system (4.32)–(4.34) reduces to

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla p = 0 \quad \text{with } \nabla \cdot b\boldsymbol{u} = 0. \tag{4.36}$$

These are the so-called "lake equations" of Ref. [10], placed into a rotating frame. Eqs. (4.36) with $f = 0$ and $\epsilon = 1$ are obtained in [10] as the leading order equations in an asymptotic expansion of the EB equations (1.5) in powers of the aspect ratio α in Eq. (1.7). See Refs. [10,21,22] for detailed discussions of the properties of the lake equations and also of the higher-order, nonhydrostatic, equations that arise in the shallow-water expansion of the EB system.

4.2.2. $O(\epsilon^2)$ *variation and higher-order fluid equations*

Next, the variation $\delta \mathcal{L}_2$ from (4.11) is calculated as follows:

$$\begin{aligned}\delta \mathcal{L}_2 &= \int \mathrm{d}t \int \mathrm{d}^3x\, [(\boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}})\delta\mathcal{D} + \mathcal{D}\boldsymbol{u}_{\mathrm{D}} \cdot \delta\boldsymbol{u} + \mathcal{D}(\boldsymbol{u}_{\mathrm{R}} - \epsilon\boldsymbol{u}_{\mathrm{D}}) \cdot \delta\boldsymbol{u}_{\mathrm{D}}] \\ &= \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\mathrm{D}} - u_{\mathrm{R}j} \nabla_3 u_{\mathrm{D}}^j + \tfrac{1}{2}\epsilon \nabla_3 |\boldsymbol{u}_{\mathrm{D}}|^2 \right)_i - \epsilon \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{D}\chi\delta w.\end{aligned} \tag{4.37}$$

Using the relations in (2.6) allows us to express the last term in (4.37) as

$$-\epsilon \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{D}\chi\delta w = -\int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{D}(\mathcal{D}^{-1})^i_A \delta l^A \left(\hat{\boldsymbol{z}} \frac{\mathrm{d}\chi}{\mathrm{d}t} + \epsilon \chi \nabla_3 w \right)_i. \tag{4.38}$$

The motion equation arising from HP, $\delta(\mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2) = 0$, is found to be

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \hat{\boldsymbol{z}}\rho + \nabla_3 \left(p + \tfrac{1}{2}\epsilon^3 |\boldsymbol{u}_{\mathrm{D}}|^2\right) - \epsilon^3 \hat{\boldsymbol{z}} \frac{\mathrm{d}\chi}{\mathrm{d}t} - \epsilon^4 \chi \nabla_3 w = 0. \tag{4.39}$$

This equation retains rapid fluctuations due to waves of horizontal compression at order $O(\epsilon^2)$ and is not balanced in our sense, because its 3-divergence contains time derivatives. Moreover, it differs from the PE only by $O(\epsilon^3)$, since the term $\epsilon^2 u_{\mathrm{R}j} \nabla_3 u_{\mathrm{D}}^j$ in (4.16) *cancels* in computing (4.39), as might be anticipated from the "nested" appearance of the sum (4.11) for the action. We pursue Eq. (4.39) no further. Its only purpose is to show which terms are being neglected in the $\epsilon \ll 1$ asymptotics of HP for the PE, when we truncate at order $O(\epsilon^2)$ in Eq. (4.16). Varying the fourth term in the Rossby-number series for the action in (4.11) cancels the $O(\epsilon^3)$ and $O(\epsilon^4)$ quantities in (4.39) and returns us to the PE.

Prospect. For the rest of this paper, we shall focus our attention on the HBE (4.16)–(4.18). Obtained systematically from a Rossby-number expansion of HP for the PE, the HBE system inherits several advantages beyond that of being balanced. These advantages include self-consistent boundary conditions, Kelvin's theorem, and conservation of energy and potential vorticity, as well as a Hamiltonian formulation in which to make at least qualitative comparisons between its solution properties and the corresponding properties of the PE which it approximates.

5. Comparison of the HBE with PE and other BEs in the literature

The HBE (4.16)–(4.18) may be conveniently rewritten in terms of horizontal and vertical velocity components, as

$$\begin{aligned}
&\epsilon\partial_t \boldsymbol{u}_{\mathrm{R}} + \epsilon^2 w \boldsymbol{u}_{\mathrm{R}z} + (\epsilon\zeta + f)\hat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla\big(p + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 + \epsilon^2 \boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}}\big) = 0,\\
&\rho + p_z + \epsilon^2 \boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}z} = 0,\\
&\quad \text{with } \frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t\rho + \boldsymbol{u}_3\cdot\nabla_3\rho = \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho + \epsilon w\rho_z = 0,\\
&\qquad \nabla\cdot\boldsymbol{u} + \epsilon w_z = 0, \quad \text{where } \boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon\boldsymbol{u}_{\mathrm{D}}.
\end{aligned} \tag{5.1}$$

The HBE differ from the PE at order $O(\epsilon^2)$. The nature of this difference becomes clear, upon writing the PE (3.1) as "HBE plus order $O(\epsilon^2)$",

$$\begin{aligned}
&\epsilon\partial_t \boldsymbol{u}_{\mathrm{R}} + \epsilon^2 w \boldsymbol{u}_{\mathrm{R}z} + (\epsilon\zeta + f)\hat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla\big(p + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 + \epsilon^2 \boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}}\big)\\
&\quad = -\epsilon^2\partial_t\boldsymbol{u}_{\mathrm{D}} - \epsilon^3 w\boldsymbol{u}_{\mathrm{D}z} - \tfrac{1}{2}\epsilon^3\nabla|\boldsymbol{u}_{\mathrm{D}}|^2,\\
&\rho + p_z + \epsilon^2 \boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}z} = \epsilon^2 \boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}z},\\
&\quad \text{with } \frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho + \epsilon w\rho_z = 0,\\
&\qquad \nabla\cdot\boldsymbol{u} + \epsilon w_z = 0, \quad \text{where } \boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon\boldsymbol{u}_{\mathrm{D}},
\end{aligned} \tag{5.2}$$

in which the left-hand sides of the first two equations are the HBE, and the right-hand sides are order $O(\epsilon^2)$ adjustments required to obtain the PE. We interpret the $O(\epsilon^2)$ differences as unbalanced perturbations of HBE that could potentially lead to high-frequency, or rapidly growing fluctuations [9].

Expanding the buoyancy ρ around a time-independent background stratification $\overline{\rho}(z)$ as $\rho = \overline{\rho}(z) + \epsilon\hat{\rho}(\boldsymbol{x}_3, t)$ gives the buoyancy equation as

$$\textit{Buoyancy equation}: \quad \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} - S(z)w = 0, \quad \text{where } -S(z) = \frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z}. \tag{5.3}$$

By taking the curl of the horizontal motion equation in (5.1), the equation for the vertical component of vorticity, $\zeta = \hat{\boldsymbol{z}}\cdot\mathrm{curl}\,\boldsymbol{u}_{\mathrm{R}} = \Delta\psi$, is found to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\zeta + \frac{f}{\epsilon}\right) + (\epsilon\zeta + f)\,\Delta\chi + \epsilon\nabla w\cdot\nabla\psi_z = 0, \tag{5.4}$$

or, equivalently,

$$\textit{Vorticity equation}: \quad \partial_t \zeta + [\psi, \zeta + f/\epsilon] + f\Delta\chi + \epsilon\nabla\cdot(w\nabla\psi_z + \zeta\nabla\chi) = 0, \tag{5.5}$$

where the square brackets denote the Jacobian, e.g., $[\psi, \chi] = \psi_x\chi_y - \psi_y\chi_x$. Now $|\nabla f| = O(\epsilon)$, by the scaling assumption (1.9). So $\boldsymbol{u}\cdot\nabla f/\epsilon = O(1)$ and the ζ equation (5.4) contains only O(1) and $O(\epsilon)$ terms. Taking the horizontal divergence of the $\boldsymbol{u}_{\mathrm{R}}$ equation in (5.1) gives the following balanced equation for pressure (in which we recall that $\boldsymbol{u}_{\mathrm{R}} = \hat{z}\times\nabla\psi$, $\boldsymbol{u}_{\mathrm{D}} = \nabla\chi$, and $w_z = -\Delta\chi$),

$$\begin{aligned}\textit{Divergence equation}: \quad & \Delta p - \nabla\cdot(f\nabla\psi) - 2\epsilon[\psi_x, \psi_y] + \epsilon[\chi, f] \\ & +\epsilon^2(\Delta[\psi, \chi] + [\chi, \Delta\psi] + [\psi_z, w]) = 0,\end{aligned} \tag{5.6}$$

The divergence equation (5.6) relates pressure and velocity to each other without requiring any time derivatives. Hence, these equations are balanced. The z component of the motion equation in (5.1) is also balanced in the same sense,

$$\textit{Hydrostasy equation}: \quad \rho + p_z + \epsilon^2[\psi, \chi_z] = 0. \tag{5.7}$$

In comparison to other BE in the literature, the BE discussed in [12,13] retain only order O(1) and order $O(\epsilon)$ terms in Eqs. (5.4)–(5.7) and in their boundary conditions. The BEM equations of Allen [3] differ from Eqs. (5.4)–(5.7), by having one different $O(\epsilon^2)$ term in the divergence equation (5.6) and *no* $O(\epsilon^2)$ term appearing in the hydrostasy equation (5.7). The $O(\epsilon^2)$ term in the HBE (5.7) breaks the exact hydrostatic relationship of the PE, in order to ensure that the balanced motion equations (5.1) will be Hamiltonian in the original Eulerian fluid variables. (This order $O(\epsilon^2)$ term will also play an important role in deriving a balanced omega equation for w.) As pointed out to the author by P. Gent, the HBE equations transform to the BEM equations of Allen [3], upon introducing a new pressure, $\mathcal{P}$, related to the physical pressure, p, and horizontal velocity decomposition $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon\boldsymbol{u}_{\mathrm{D}}$, by

$$\mathcal{P}_z = p_z + \epsilon^2\boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}z} = p_z + \epsilon^2[\psi, \chi_z], \tag{5.8}$$

which may be formally integrated, as

$$p + \epsilon^2\boldsymbol{u}_{\mathrm{R}}\cdot\boldsymbol{u}_{\mathrm{D}} = \mathcal{P} + \epsilon^2\int^z \boldsymbol{u}_{\mathrm{R}z}\cdot\boldsymbol{u}_{\mathrm{D}}\,\mathrm{d}z, \tag{5.9}$$

where p is the pressure in our notation and $\mathcal{P}$ is the pressure in the notation of Ref. [3]. In the rest of this paper, we shall retain our present notation for pressure. However, we keep in mind the equivalence of the HBE equations to Allen's BEM model [3] under Gent's redefinition of pressure (5.9), especially for the sake of the numerical solution procedures and linearized analysis discussed in [3]. This equivalence between HBE and BEM also allows us to take advantage of the numerical and analytical comparisons already made between solutions of BEM and other balanced models [3,7]. These numerical comparisons are made with horizontally periodic boundary conditions [7]; so the difference in the definitions of pressure in (5.9) between HBE and BEM in the horizontal boundary condition (4.23) is not an issue. Also, the curvature κ at the boundary vanishes for a periodic domain.

Although the HBE divergence and hydrostasy equations (5.6) and (5.7) for p and χ decouple from the rest of the system and could be addressed separately, it is better to solve them as part of the larger system (5.1). The method for doing this in BE theory is to calculate a so-called "omega equation", [12]. The omega equation for HBE is found by calculating the sum of three equations: the Laplacian Δ operating on the buoyancy equation (5.3), plus the operator $f\partial/\partial z$ acting on the vorticity equation (5.5) and $\partial^2/\partial z\partial t$ acting on the divergence equation (5.6). On taking this sum and using the hydrostasy equation (5.7), the highest order derivative terms *cancel identically* and one finds the following omega equation for determining the vertical velocity w,

$$\textit{Omega equation}: \quad -S(z)\Delta w - f^2 w_{zz} + \mathrm{LOT} = 0, \tag{5.10}$$

where the terms denoted LOT are lower order in derivatives of w. Thus, the solution for the vertical velocity w in HBE is obtained from an elliptic equation. The appropriate horizontal boundary condition for w in a finite domain comes from tangency of the horizontal velocity, the buoyancy equation (5.3) and hydrostasy equation (5.7). This boundary condition is given by

$$w = \frac{-1}{\mathcal{P}_{zz}}\,(\partial_t \mathcal{P}_z + (\psi_n + \epsilon\chi_s)\partial_s \mathcal{P}_z) \quad \text{on the boundary}, \tag{5.11}$$

where $\mathcal{P}$ is given in Eq. (5.8). The vertical boundary conditions for w are given in (1.10). Numerical solution procedures for BEM are described in detail in [3]. These same numerical procedures apply for HBE, modulo the redefinition in pressure p to $\mathcal{P}$ given in Eq. (5.8). In addition, because the omega equation (5.10) for HBE is elliptic, standard methods for BE also apply in numerical solutions of HBE.

6. Kelvin circulation theorem, potential-vorticity advection and energy conservation for HBE

The HBE (4.16) implies the following Kelvin circulation theorem for HBE:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} (\boldsymbol{R} + \epsilon\boldsymbol{u}_{\mathrm{R}})\cdot \mathrm{d}\boldsymbol{x}_3 &= \oint_{\gamma(t)} \left[\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R} + \epsilon\boldsymbol{u}_{\mathrm{R}}) + (R_j + \epsilon u_{\mathrm{R}j})\boldsymbol{\nabla}_3 u_3^j\right]\cdot \mathrm{d}\boldsymbol{x}_3 \\ &= -\oint_{\gamma(t)} \rho\hat{\boldsymbol{z}}\cdot \mathrm{d}\boldsymbol{x}_3 + \oint_{\gamma(t)} \boldsymbol{\nabla}_3\big(-p + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 + \boldsymbol{u}\cdot\boldsymbol{R}\big)\cdot \mathrm{d}\boldsymbol{x}_3 \\ &= -\oint_{\gamma(t)} \rho\,\mathrm{d}z\end{aligned} \tag{6.1}$$

for any closed curve $\gamma(t)$ moving with the fluid. We compare this result with the Kelvin circulation theorem for PE in Eq. (3.5), rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \underbrace{(\boldsymbol{R} + \epsilon\boldsymbol{u})}_{\text{PE}}\cdot \mathrm{d}\boldsymbol{x}_3 = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \underbrace{(\boldsymbol{R} + \epsilon\boldsymbol{u}_{\mathrm{R}}}_{\text{HBE}} + \underbrace{\epsilon^2\boldsymbol{u}_{\mathrm{D}})\cdot \mathrm{d}\boldsymbol{x}_3}_{\text{Zero}} = -\oint_{\gamma(t)} \rho\,\mathrm{d}z. \tag{6.2}$$

Since the last term vanishes, by $\oint \boldsymbol{u}_{\mathrm{D}}\cdot \mathrm{d}\boldsymbol{x}_3 = \oint \mathrm{d}\chi = 0$, one expects the HBE circulation integral to differ from that of PE only through the differences in buoyancy between the two theories.

6.1. Potential vorticity for HBE

Applying Stokes theorem to (6.1) on a surface of constant buoyancy ρ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int_{S(t)|_\rho} [\mathrm{curl}_3(\epsilon\boldsymbol{u}_{\mathrm{R}} + \boldsymbol{R})\cdot \boldsymbol{\nabla}_3\rho]\,\frac{\mathrm{d}S}{|\boldsymbol{\nabla}_3\rho|} = 0. \tag{6.3}$$

Thus, the flux of total vorticity through a surface of constant buoyancy is invariant for HBE. Taking the three-dimensional curl of the $\boldsymbol{u}_{\mathrm{R}}$-equation in the HBE (4.16), rewritten as

$$\epsilon \partial_t \boldsymbol{u}_{\mathrm{R}} - \boldsymbol{u}_3 \times \operatorname{curl}_3(\epsilon \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{R}) + \rho \hat{\boldsymbol{z}} + \boldsymbol{\nabla}_3\big(p + \tfrac{1}{2}\epsilon |\boldsymbol{u}_{\mathrm{R}}|^2 + \epsilon^2 \boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}}\big) = 0, \tag{6.4}$$

implies the total-vorticity relation

$$\partial_t \operatorname{curl}_3(\epsilon \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{R}) - \operatorname{curl}_3\,(\boldsymbol{u} \times \operatorname{curl}_3(\epsilon \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{R})) = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_3 \rho. \tag{6.5}$$

As a consequence of Eq. (6.5) and $\mathrm{d}\rho/\mathrm{d}t = 0$, the potential vorticity, Q, satisfies the advection law, cf. Eq. (2.16),

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \boldsymbol{u}_3 \cdot \boldsymbol{\nabla}_3 Q = 0, \quad \text{where } Q \equiv \operatorname{curl}_3(\epsilon \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{R}) \cdot \boldsymbol{\nabla}_3 \rho. \tag{6.6}$$

Advection of both Q and ρ, combined with the tangential boundary conditions on $\boldsymbol{u}_3$, yields an infinity of conserved quantities,

$$C_\Phi = \int \mathrm{d}^3x\, \Phi(Q, \rho), \tag{6.7}$$

for any function Φ.

6.2. HBE energy

An energy conservation equation may also be derived for the HBE, by taking the scalar product of $\boldsymbol{u}_3 = (\boldsymbol{u}, \epsilon w)$ with Eq. (6.4), summing the result with z times $\mathrm{d}\rho/\mathrm{d}t = 0$, and using $\boldsymbol{\nabla}_3 \cdot \boldsymbol{u}_3 = 0$ to find

$$\frac{\partial}{\partial t}\big(\tfrac{1}{2}\epsilon |\boldsymbol{u}_{\mathrm{R}}|^2 + z\rho\big) = -\epsilon^2 \boldsymbol{\nabla} \cdot (\chi \partial_t \boldsymbol{u}_{\mathrm{R}}) - \boldsymbol{\nabla}_3 \cdot \big(p + z\rho + \tfrac{1}{2}\epsilon |\boldsymbol{u}_{\mathrm{R}}|^2 + \epsilon^2 \boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}}\big)\boldsymbol{u}_3. \tag{6.8}$$

The integrated energy,

$$E = \int \mathrm{d}^3x \big(\tfrac{1}{2}\epsilon |\boldsymbol{u}_{\mathrm{R}}|^2 + z\rho\big), \tag{6.9}$$

is then time-independent as a consequence of the boundary conditions (1.10) and (4.10), namely

$$\boldsymbol{u}_3 \cdot \hat{\boldsymbol{n}}_3 = 0 \quad \text{and} \quad \hat{\boldsymbol{n}} \cdot \partial_t \boldsymbol{u}_{\mathrm{R}} = 0 \quad \text{at the boundary}. \tag{6.10}$$

The conserved quantities E and C_Φ are evidently the asymptotic legacy of the corresponding quantities for the PE and result from the derivation of HBE from HP. More of the nature of these conserved quantities will be clear from the Hamiltonian formulation of the HBE given in Section 7.

7. Hamiltonian formulation of Eulerian HBE

7.1. Legendre transformation in Eulerian variables

In order to find its associated Hamiltonian, we Legendre transform the HBE action, (4.11) at order $\mathrm{O}(\epsilon)$. The action at this order is given by

$$\begin{aligned} \mathcal{L}_{\mathrm{HBE}} &= \mathcal{L}_0 + \epsilon \mathcal{L}_1 \\ &= \int \mathrm{d}t \int \mathrm{d}^3x [\mathcal{D}\boldsymbol{u} \cdot \boldsymbol{R}(\boldsymbol{x}) - \mathcal{D}\rho z - p(\mathcal{D} - 1)] + \epsilon \int \mathrm{d}t \int \mathrm{d}^3x\, \tfrac{1}{2}\mathcal{D}|\boldsymbol{u} - \epsilon \boldsymbol{u}_{\mathrm{D}}|^2. \end{aligned} \tag{7.1}$$

The momentum π_A canonically conjugate to l^A at this order in ϵ may be computed using the variational formulas in Eqs. (2.6) as follows:

$$\begin{aligned}\pi_A &= \frac{\delta \mathcal{L}_{\mathrm{HBE}}}{\delta l^A_{,t}} = \frac{\delta u_3^i}{\delta l^A_{,t}} \frac{\delta \mathcal{L}_{\mathrm{HBE}}}{\delta u_3^i} \\ &= -(\mathcal{D}^{-1})^i_A \frac{\delta \mathcal{L}_{\mathrm{HBE}}}{\delta u_3^i} \\ &= -(D^{-1})^i_A \mathcal{D}(R_i + \epsilon u_{\mathrm{R}i}) + \epsilon (D^{-1})^3_A \Delta^{-1} \partial_z \partial_j \mathcal{D} u^j_{\mathrm{R}},\end{aligned} \tag{7.2}$$

where we have used $\chi = -\Delta^{-1} w_z$ and integrated by parts in calculating the last term. The momentum density defined by $m_i = -\pi_A D^A_i$ with $i = 1, 2, 3$, is given by

$$m_i = \mathcal{D}(R_i + \epsilon u_{\mathrm{R}i}) \quad \text{for } i = 1, 2, \qquad m_3 = -\epsilon \Delta^{-1} (\mathcal{D} u^j_{\mathrm{R}})_{,jz} \quad \text{for } i = 3. \tag{7.3}$$

Eventually, m_3 will vanish when the condition $(\mathcal{D} u^j_{\mathrm{R}})_{,j} = 0$ is imposed. However, we must first take variations of the HBE Hamiltonian with respect to m_3. Using (7.2) and the definition of the fluid velocity in Eq. (2.3) leads to a useful formula

$$\pi_A l^A_{,t} = \boldsymbol{m}_3 \cdot \boldsymbol{u}_3 = \boldsymbol{m} \cdot \boldsymbol{u} + \epsilon m_3 w. \tag{7.4}$$

The Legendre transformation to the Hamiltonian for HBE is then expressed purely in terms of Eulerian variables, as

$$\begin{aligned}H_{\mathrm{BE}} &= \int \mathrm{d}^3x \big[\pi_A l^A_{,t} - L_{\mathrm{HBE}}\big] \\ &= \int \mathrm{d}^3x \big[\boldsymbol{m}_3 \cdot \boldsymbol{u}_3 - \mathcal{D}\boldsymbol{u} \cdot \boldsymbol{R} - \tfrac{1}{2}\epsilon \mathcal{D} |\boldsymbol{u} - \epsilon \boldsymbol{u}_{\mathrm{D}}|^2 + \mathcal{D} z \rho + p(\mathcal{D} - 1)\big].\end{aligned} \tag{7.5}$$

Substituting the definitions $\boldsymbol{m} = \mathcal{D}(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{R}})$ and $\boldsymbol{u}_{\mathrm{R}} = \boldsymbol{u} - \epsilon \boldsymbol{u}_{\mathrm{D}}$ into Eq. (7.5) gives the Hamiltonian

$$H_{\mathrm{BE}} = \int \mathrm{d}^3x \left[\frac{1}{2\epsilon \mathcal{D}} |\boldsymbol{m} - \mathcal{D}\boldsymbol{R}|^2 + \mathcal{D} z \rho + p(\mathcal{D} - 1) + \epsilon w m_3 + \epsilon (\boldsymbol{m} - \mathcal{D}\boldsymbol{R}) \cdot \boldsymbol{u}_{\mathrm{D}}\right], \tag{7.6}$$

which consists of the sum of the order $O(\epsilon)$ approximate kinetic energy, the potential energy, and a sum of terms that appear in the form of constraints on H_{BE}. The variational derivatives of $H_{\mathrm{BE}}(\boldsymbol{m}_3, \mathcal{D}, \rho)$ are given by

$$\begin{aligned}\delta H_{\mathrm{BE}} = \int \mathrm{d}^3x \big[&\boldsymbol{u}_3 \cdot \delta \boldsymbol{m}_3 + \mathcal{D} z \delta \rho + \big(-\tfrac{1}{2}\epsilon |\boldsymbol{u}_{\mathrm{R}}|^2 - \boldsymbol{u} \cdot \boldsymbol{R} + z\rho + p\big)\delta \mathcal{D} \\ &+ (\mathcal{D} - 1)\delta p + m_3 \delta w + [\boldsymbol{m} - \mathcal{D}(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{R}})] \cdot \delta \boldsymbol{u} \\ &+ \delta\chi \boldsymbol{\nabla} \cdot (\mathcal{D}\boldsymbol{u}_{\mathrm{R}})\big] - \int \mathrm{d}z \oint \mathrm{d}s\, \mathcal{D} \delta \chi \hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{R}}.\end{aligned} \tag{7.7}$$

Thus, the quantities p, w, χ and $\boldsymbol{u}$ appear to be five Lagrange multipliers which impose the following constraints:

$$\begin{aligned}&\delta p: \quad \mathcal{D} - 1 = 0, \quad \text{volume preservation of the flow;} \\ &\delta \boldsymbol{u}: \quad \boldsymbol{m} - \mathcal{D}(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{R}}) = 0, \quad \text{which defines } \boldsymbol{m} \text{ in terms of } \boldsymbol{u}_{\mathrm{R}} \text{ and } \mathcal{D}; \\ &\delta w: \quad m_3 = 0, \text{ as expected with } m_3 = -\epsilon \Delta^{-1} \partial_z \boldsymbol{\nabla} \cdot \mathcal{D}\boldsymbol{u}_{\mathrm{R}}; \\ &\delta \chi: \quad \boldsymbol{\nabla} \cdot (\mathcal{D}\boldsymbol{u}_{\mathrm{R}}) = 0 \text{ in the interior and } \boldsymbol{u}_{\mathrm{R}} \cdot \hat{\boldsymbol{n}} = 0 \text{ at the boundary.}\end{aligned} \tag{7.8}$$

However, the variations $\delta\chi$ and δw are linked by the formula

$$\delta\chi = -\Delta^{-1}\partial_z\delta w. \tag{7.9}$$

Hence, we may integrate by parts in Eq. (7.7) to regroup these terms, as

$$\begin{aligned}\delta H_{\mathrm{BE}} = \int \mathrm{d}^3x\big[&\boldsymbol{u}_3\cdot\delta\boldsymbol{m}_3 + \mathcal{D}z\delta\rho + \big(-\tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 - \boldsymbol{u}\cdot\boldsymbol{R} + z\rho + p\big)\delta\mathcal{D}\\ &+(\mathcal{D}-1)\delta p + [\boldsymbol{m} - \mathcal{D}(\boldsymbol{R}+\epsilon\boldsymbol{u}_{\mathrm{R}})]\cdot\delta\boldsymbol{u}\\ &+(m_3 + \Delta^{-1}\partial_z\boldsymbol{\nabla}\cdot(\boldsymbol{m}-\mathcal{D}\boldsymbol{R}))\delta w\big] - \oint \mathrm{d}s\,\delta\chi\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{R}}.\end{aligned} \tag{7.10}$$

Thus, vanishing of the variations of H_{BE} with respect to the velocity components (the $\delta\boldsymbol{u}$ and δw terms) defines the components of the momentum density, cf. Eq. (7.3). This is a general property of the Legendre transformation [6]. This property also holds, e.g., in the case of the Hamiltonian formulation of the PE, which is the same as for HBE modulo terms of order $O(\epsilon^2)$.

Remark. The constrained Hamiltonian H_{BE} is equal to the conserved energy E in (6.9) when the constraints in (7.8) are imposed. Actually H_{BE} is a Routhian; the pressure p is not Legendre-transformed, since it has no canonically conjugate momentum. See, e.g., [19] for the analogous situation in the case of the incompressible Euler equations.

7.2. Lie–Poisson bracket in Eulerian variables

The change of variables from π_A and l^A to $\boldsymbol{m}_3$, ρ and $\mathcal{D}$, given by

$$\boldsymbol{m}_3 = -\pi_A\boldsymbol{\nabla}_3 l^A, \qquad \rho = \rho(l^A), \qquad \mathcal{D} = \det(\boldsymbol{\nabla}_3 l^A), \tag{7.11}$$

transforms the canonical Poisson bracket arising from HP, namely,

$$\{F,G\}(\pi_A, l^A) = -\int \mathrm{d}^3x\left[\frac{\delta F}{\delta\pi_A}\frac{\delta G}{\delta l^A} - \frac{\delta G}{\delta\pi_A}\frac{\delta F}{\delta l^A}\right], \tag{7.12}$$

into the following Lie–Poisson bracket in terms of variables $\boldsymbol{m}_3$, ρ and D, whose algebraic properties are discussed in full detail in [16,17,19,20] and references therein

$$\begin{aligned}\{F,G\}(\boldsymbol{m}_3,\rho,\mathcal{D}) = -\int \mathrm{d}^3x\bigg[&\frac{\delta F}{\delta m_{3i}}\left((\partial_j m_{3i} + m_{3j}\partial_i)\frac{\delta G}{\delta m_{3j}} - \rho_{,i}\frac{\delta G}{\delta\rho} + \mathcal{D}\partial_i\frac{\delta G}{\delta\mathcal{D}}\right)\\ &+\frac{\delta F}{\delta\rho}\rho_{,j}\frac{\delta G}{\delta m_{3j}} + \frac{\delta F}{\delta\mathcal{D}}\partial_j\mathcal{D}\frac{\delta G}{\delta m_{3j}}\bigg],\end{aligned} \tag{7.13}$$

where $\partial_j = \partial/\partial x^j$, $j = 1,2,3$, operates on all terms it multiplies to its right. This Lie–Poisson bracket satisfies the Jacobi identity,

$$\{E,\{F,G\}\} + \{F,\{G,E\}\} + \{G,\{E,F\}\} = 0, \tag{7.14}$$

for any functionals E, F and G of $\boldsymbol{m}_3$, ρ and $\mathcal{D}$, simply because (7.14) is a variable transform of the Jacobi identity for the canonical Poisson bracket.

The corresponding equations of motion are given in Lie–Poisson Hamiltonian form by

$$\begin{aligned}&\partial_t m_{3i} = \{m_{3i}, H_{\mathrm{BE}}\} = -(\partial_j m_{3i} + m_{3j}\partial_i)u_3^j + \rho_{,i}\mathcal{D}z - \mathcal{D}\partial_i\big(-\tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 - \boldsymbol{u}\cdot\boldsymbol{R} + z\rho + p\big),\\ &\partial_t\rho = \{\rho, H_{\mathrm{BE}}\} = -\rho_{,j}u_3^j, \qquad \partial_t\mathcal{D} = \{\mathcal{D}, H_{\mathrm{BE}}\} = -\partial_j\mathcal{D}u_3^j.\end{aligned} \tag{7.15}$$

These are the HBE (4.16) now expressed in terms of $\boldsymbol{m}_3$, ρ and D. Substituting into (7.15) the definitions of the components of m_{3i} given in Eq. (7.3) recovers the HBE motion equation in the form that appears in the Kelvin theorem calculation in (6.1), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{R}) + (\epsilon u_{\mathrm{R}j} + R_j)\boldsymbol{\nabla}_3 u_3^j = -\rho\hat{\boldsymbol{z}} + \boldsymbol{\nabla}_3\big(-p + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 + \boldsymbol{u}\cdot\boldsymbol{R}\big). \tag{7.16}$$

If $\mathcal{D}$ is initially equal to unity, it will remain so under the dynamics of (7.15), provided $\boldsymbol{\nabla}_3 \cdot \boldsymbol{u}_3 = 0$. If $m_3 = 0$, it will remain so under the dynamics of (7.15), provided the modified hydrostatic condition $\rho + p_z + \epsilon^2 \boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}z} = 0$ is satisfied. The Lagrange multipliers p and χ satisfy the divergence and hydrostasy equations (5.6) and (5.7).

Remarks. (1) The Hamiltonian (7.6) for HBE differs from that of PE by terms of order $\mathrm{O}(\epsilon^2)$, and the Hamiltonian structures of the two theories are identical in form, but with $\mathrm{O}(\epsilon^2)$ differences in the expressions for their horizontal momentum densities. See Holm and Long [18] and Roulstone and Brice [29]. In both theories, the constraints imposed by the Lagrange multipliers p (for volume preservation) and w (for hydrostasy) each remove one degree of freedom from EB. In HBE, the relation between w and χ removes one more, so only two degrees of freedom remain, ψ and ρ.

(2) The mathematical nature of the Lie–Poisson bracket (7.13) for HBE and others like it (which are defined on spaces dual to semidirect-product Lie algebras) is discussed in detail in [16,17,19,20]. The Casimirs C_Φ for the Lie–Poisson bracket (7.13) have variational derivatives that lie in the null space of the Poisson operator for (7.13) and, thus, they Poisson-commute with *every* functional of the Eulerian variables $\boldsymbol{m}_3$, ρ and $\mathcal{D}$. Consequently, they are conserved, since they Poisson-commute with the Hamiltonian H_{BE}, which is expressed in Eulerian variables and generates the HBE evolution under the operation of Lie–Poisson bracket. One particular implication of the Casimirs is that the canonical transformations they generate in the Lagrangian variables l^A are the steady HBE flows. Thus, the steady HBE flows are *relative* equilibria of the Casimirs. That is, they are critical points of the sum $H_{\mathrm{C}} = H_{\mathrm{BE}} + C_\Phi$ of the Hamiltonian H_{BE} and the Casimirs, C_Φ. The functional freedom in the definition of the Casimirs allows *all* of the steady HBE flows to be characterized this way. This setting makes it possible to investigate the Lyapunov stability properties of the HBE steady flows using the methods described in [2,18,20]. We do not pursue such a stability investigation for HBE steady flows here, because of its $\mathrm{O}(\epsilon^2)$ similarity to that for the PE, already discussed in [18].

8. Isopycnal representation of PE

In the PE (3.1) the buoyancy ρ is a Lagrangian coordinate, which may be used in place of the vertical Eulerian coordinate z, provided the function

$$z = h(x, y, \rho, t) \tag{8.1}$$

is one-to-one (i.e., provided $h_\rho = 1/\rho_z \neq 0$). Upon making that assumption, the action $\mathcal{L}_{\mathrm{PE}}$ in HP for the PE in Eq. (3.14) transforms to

$$\mathcal{L}_{\mathrm{PE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\big[\tfrac{1}{2}\epsilon\tilde{D}|\tilde{\boldsymbol{u}}|^2 + \tilde{D}\tilde{\boldsymbol{u}}\cdot\boldsymbol{R}(\boldsymbol{x}) - \tilde{D}\rho h - \tilde{p}(\tilde{D} - h_\rho)\big], \tag{8.2}$$

where tilde $\sim$ denotes dependence on (x, y, ρ, t). For example, $\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v})$ are the components in the fixed x and y directions of the horizontal fluid velocity, expressed as a function of (x, y, ρ, t). That is,

$$\tilde{\boldsymbol{u}}(x, y, \rho, t) = \boldsymbol{u}(x, y, h(x, y, \rho, t), t). \tag{8.3}$$

Likewise, the Lagrangian coordinates $\tilde{l}^a(x, y, \rho, t)$ are related to the earlier ones, $l^a(x, y, z, t)$, $a = 1, 2$, by

$$\tilde{l}^a(x, y, \rho, t) = l^a(x, y, h(x, y, \rho, t), t), \quad a = 1, 2. \tag{8.4}$$

Thus,

$$\tilde{\nabla}\tilde{l}^a = \tilde{\nabla} l^a(x, y, h(x, y, \rho), t) = \nabla l^a + l^a_{,z}\tilde{\nabla}h, \tag{8.5}$$

where $\tilde{\nabla}$ denotes (∂_x, ∂_y) at constant ρ. By the definition (8.4), l^a and $\tilde{l}^a$ satisfy the chain-rule identities,

$$\begin{aligned} l^a_{,z} &= \frac{1}{h_\rho}\tilde{l}^a_{,\rho} \quad \text{with } a = 1, 2, \\ \nabla l^a &= \tilde{\nabla}\tilde{l}^a - \frac{1}{h_\rho}\tilde{l}^a_{,\rho}\tilde{\nabla}h, \qquad \partial_t l^a = \partial_t \tilde{l}^a - \frac{1}{h_\rho}\tilde{l}^a_{,\rho}\partial_t h. \end{aligned} \tag{8.6}$$

Being Lagrangian coordinates, the $\tilde{l}^a(x, y, \rho, t)$ also satisfy the advection rule

$$\begin{aligned} 0 &= \frac{\mathrm{d}\tilde{l}^a}{\mathrm{d}t} \equiv \partial_t \tilde{l}^a + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{l}^a \quad \text{with } a = 1, 2, \\ &= \frac{\mathrm{d}}{\mathrm{d}t} l^a(x, y, h(x, y, \rho, t), t) = \partial_t l^a + \boldsymbol{u} \cdot \nabla l^a + l^a_{,z}(\partial_t h + \boldsymbol{u} \cdot \tilde{\nabla}h). \end{aligned} \tag{8.7}$$

Hence, the vertical velocity $\tilde{w}(x, y, \rho, t) = w(x, y, h(x, y, \rho, t), t)$ is found from the last term of (8.7) to be

$$\epsilon\tilde{w}(x, y, \rho, t) = \frac{\mathrm{d}h}{\mathrm{d}t} = \partial_t h + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}h, \tag{8.8}$$

upon using definition (8.3) to replace $\boldsymbol{u}$ by $\tilde{\boldsymbol{u}}$. Note that Eq. (8.8) implies that the space and time derivatives of h are of order $\mathrm{O}(\epsilon)$, i.e.,

$$|\partial_t h| = \mathrm{O}(\epsilon) \quad \text{and} \quad |\tilde{\nabla}h| = \mathrm{O}(\epsilon). \tag{8.9}$$

In also follows from Eq. (8.8) and the incompressibility relation,

$$\nabla_3 \cdot \boldsymbol{u}_3 = \tilde{\nabla} \cdot \tilde{\boldsymbol{u}} - \frac{1}{h_\rho}\tilde{\boldsymbol{u}}_\rho \cdot \tilde{\nabla}h + \frac{\epsilon}{h_\rho}\tilde{w}_\rho, \tag{8.10}$$

that

$$\partial_t h_\rho = -\tilde{\nabla} \cdot h_\rho \tilde{\boldsymbol{u}}. \tag{8.11}$$

Finally, Eq. (8.7) for the advection of $\tilde{l}^a$ implies that the isopycnal Jacobian $\tilde{D} = \det(\tilde{\nabla}\tilde{l}^a)$ satisfies

$$\partial_t \tilde{D} = -\tilde{\nabla} \cdot \tilde{D}\tilde{\boldsymbol{u}}. \tag{8.12}$$

So $\tilde{D}$ and h_ρ satisfy the same equation. This is the meaning of the constraint imposed by $\tilde{p}$ in the action (8.2).

Varying the action $\mathcal{L}_{\mathrm{PE}}$ in (8.2) at fixed (x, y, ρ, t) gives

$$\begin{aligned} \delta\mathcal{L}_{\mathrm{PE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho \big[&(\tilde{D}\epsilon\tilde{\boldsymbol{u}} + \tilde{D}\boldsymbol{R}(\boldsymbol{x})) \cdot \delta\tilde{\boldsymbol{u}} + \big(\tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}|^2 + \tilde{\boldsymbol{u}} \cdot \boldsymbol{R}(\boldsymbol{x}) - \rho h - \tilde{p}\big)\delta\tilde{D} \\ &- \tilde{D}\rho\delta h + \tilde{p}\delta h_\rho - (\tilde{D} - h_\rho)\delta\tilde{p}\big]. \end{aligned} \tag{8.13}$$

This formula is expressible in terms of variations $\delta\tilde{l}^a$ with respect to the Lagrange coordinate $\tilde{l}^a$ on each level surface of buoyancy, ρ, by using the definitions (cf. Eqs. (2.6))

$$\delta\tilde{D} = \tilde{D}(\tilde{D}^{-1})^i_a \delta\tilde{l}^a_{,i}, \qquad \delta\tilde{u}^i = -(\tilde{D}^{-1})^i_a \tilde{u}^j \delta\tilde{l}^a_{,j} - (\tilde{D}^{-1})^i_a \delta\tilde{l}^a_{,t}, \quad i = 1, 2, \tag{8.14}$$

where partial derivatives are taken at constant ρ. We substitute these definitions into the variational formula (8.13) for $\delta\mathcal{L}_{\mathrm{PE}}$ and integrate by parts using the tangency conditions on the velocity at the boundary and requiring δh to vanish at the endpoints in ρ. Consequently, we find (cf. Eq. (2.7))

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{PE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\big\{&\delta\tilde{l}^a\big[\partial_t(\tilde{D}(\tilde{D}^{-1})^i_a(\epsilon\tilde{u}_i + R_i)) + \partial_j(\tilde{D}\tilde{u}^j(\tilde{D}^{-1})^i_a(\epsilon\tilde{u}_i + R_i))\big]\\ &+\delta\tilde{l}^a\partial_i\big[\tilde{D}(\tilde{D}^{-1})^i_a(\tilde{p} + \rho h - \tilde{\boldsymbol{u}}\cdot\boldsymbol{R} - \tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}|^2)_{,i}\big]\\ &-(\tilde{D}\rho + \tilde{p}_\rho)\delta h - (\tilde{D} - h_\rho)\delta\tilde{p}\big\}.\end{aligned} \tag{8.15}$$

Rearrangement of formula (8.15) using the continuity equation (8.12) for $\tilde{D}$ and the identities (cf. Eqs. (2.8))

$$\partial_j\tilde{D} = \tilde{D}(\tilde{D}^{-1})^i_a\partial_j\tilde{D}^a_i,\quad (\tilde{D}(\tilde{D}^{-1})^i_a)_{,i} = 0,\quad \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{D}^{-1})^i_a = \tilde{u}^i_{\,,j}(\tilde{D}^{-1})^j_a \tag{8.16}$$

for $a, i, j = 1, 2$ gives

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{PE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\Bigg\{&\tilde{D}(\tilde{D}^{-1})^i_a\delta\tilde{l}^a\left[\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_i + \tilde{u}^j(R_{i,j} - R_{j,i}) + (\tilde{p} + \rho h)_{,i}\right]\\ &-(\tilde{D}\rho + \tilde{p}_\rho)\delta h - (\tilde{D} - h_\rho)\delta\tilde{p}\Bigg\}.\end{aligned} \tag{8.17}$$

Vanishing of $\delta\mathcal{L}_{\mathrm{PE}}$ for arbitrary variations $\delta\tilde{l}^a$, δh and $\delta\tilde{p}$ within the domain of flow implies the dimensionless isopycnal PE, called IPE,

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{u}} + f\hat{\boldsymbol{z}}\times\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\nabla}}(\tilde{p} + \rho h) = 0, \qquad \rho h_\rho + \tilde{p}_\rho = 0, \tag{8.18}$$

where we have used $\tilde{D} = h_\rho$ and $\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\hat{\boldsymbol{n}}|_\rho = 0$ at the boundary. This is a closed system, when augmented by the kinematic conditions,

$$\begin{aligned}&\epsilon\tilde{w} = \frac{\mathrm{d}h}{\mathrm{d}t} = \partial_t h + \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}h \quad &&\text{(the definition of vertical velocity)},\\ &\partial_t h_\rho = -\tilde{\boldsymbol{\nabla}}\cdot h_\rho\tilde{\boldsymbol{u}} &&\text{(incompressibility)}.\end{aligned} \tag{8.19}$$

in which $|\partial_t h|$ and $|\tilde{\boldsymbol{\nabla}}h|$ are order $\mathrm{O}(\epsilon)$. Kelvin's theorem for IPE (8.18) is expressed as, cf. Eq. (6.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\tilde{\gamma}(t)}(\boldsymbol{R} + \epsilon\tilde{\boldsymbol{u}})\cdot\mathrm{d}\boldsymbol{x} = 0, \tag{8.20}$$

where the contour $\tilde{\gamma}(t)$ moves with horizontal velocity $\tilde{\boldsymbol{u}}$ on an isopycnal surface.

Thus, the isopycnal PE (8.18) emerge from HP, upon transforming the action $\mathcal{L}_{PE}$ from Eulerian coordinates (x, y, z) to mixed Eulerian (x, y) and Lagrangian (ρ) coordinates and assuming that the Jacobian h_ρ does not vanish. The same isopycnal PE can, of course, also be obtained by applying this transformation directly to the Eulerian PE. However, we shall find this approach via the transformation of HP useful in Section 9, in constructing an alternative to the direct isopycnal representation of the Eulerian HBE derived in Section 4.

9. HP for isopycnal HBE

9.1. Recap of Eulerian HBE

The HBE in Eulerian coordinates (x, y, z) are

$$\begin{aligned}&\epsilon\partial_t \boldsymbol{u}_{\mathrm{R}} + \epsilon^2 w \boldsymbol{u}_{\mathrm{R}z} + (\epsilon\zeta + f)\hat{\boldsymbol{z}} \times \boldsymbol{u} + \boldsymbol{\nabla}\big(p + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 + \epsilon^2 \boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}}\big) = 0,\\ &\rho + p_z + \epsilon^2 \boldsymbol{u}_{\mathrm{R}} \cdot \boldsymbol{u}_{\mathrm{D}z} = 0, \quad \text{in which } \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0.\end{aligned} \tag{9.1}$$

The divergence equation (5.6) which determines the balanced pressure is rewritten here as

$$\Delta p - \boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla}\psi) - 2\epsilon[\psi_x, \psi_y] + \epsilon^2\left([\chi, f/\epsilon] + \Delta[\psi, \chi] + [\chi, \Delta\psi] + [\psi_z, w]\right) = 0. \tag{9.2}$$

Also recall that $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}} + \epsilon \boldsymbol{u}_{\mathrm{D}}$, with $\boldsymbol{u}_{\mathrm{R}} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi$, $\boldsymbol{u}_{\mathrm{D}} = \boldsymbol{\nabla}\chi$, and $w_z = -\Delta\chi$. The action in HP for the Eulerian HBE is given by $\mathcal{L}_0 + \epsilon\mathcal{L}_1$ in Eq. (4.11) as

$$\mathcal{L}_{\mathrm{HBE}} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 = \int \mathrm{d}t \int \mathrm{d}^3x\big[\tfrac{1}{2}\epsilon\mathcal{D}|\boldsymbol{u} - \epsilon\boldsymbol{u}_{\mathrm{D}}|^2 + \mathcal{D}\boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - \mathcal{D}\rho z - p(\mathcal{D} - 1)\big]. \tag{9.3}$$

9.2. Direction transformation of HBE to isopycnal coordinates

The Lagrangian coordinate ρ may be used in place of the Eulerian vertical coordinate z, provided the function

$$z = h(x, y, \rho, t) \tag{9.4}$$

is one-to-one (i.e., provided $h_\rho = 1/\rho_z \neq 0$), which we assume. The kinematics and notation of the transformation to isopycnal variables is explained in Section 8. The Helmholtz deomposition (4.2) transforms to $\tilde{\boldsymbol{u}}' = \tilde{\boldsymbol{u}}'_{\mathrm{R}} + \epsilon\tilde{\boldsymbol{u}}'_{\mathrm{D}}$, in the isopycnal representation, with $\tilde{\boldsymbol{u}}'_{\mathrm{R}}$ and $\tilde{\boldsymbol{u}}'_{\mathrm{D}}$ defined by the formulae

$$\begin{aligned}\tilde{\boldsymbol{u}}'_{\mathrm{R}} &= \hat{\boldsymbol{z}} \times \left(\tilde{\boldsymbol{\nabla}}\tilde{\psi}' - \frac{1}{h_\rho}\tilde{\psi}'_\rho\tilde{\boldsymbol{\nabla}}h\right),\\ \tilde{\boldsymbol{u}}'_{\mathrm{D}} &= \tilde{\boldsymbol{\nabla}}\tilde{\chi}' - \frac{1}{h_\rho}\tilde{\chi}'_\rho\tilde{\boldsymbol{\nabla}}h.\end{aligned} \tag{9.5}$$

Here tilde $\sim$ represents dependence on (x, y, ρ, t) and prime $'$ denotes the results of directly transforming the Eulerian Helmholtz decomposition (4.2) into isopycnal coordinates. The Eulerian HBE (9.1) then transform into isopycnal coordinates as

$$\begin{aligned}&\epsilon\partial_t\tilde{\boldsymbol{u}}'_{\mathrm{R}} + (\epsilon\tilde{\zeta}' + f)\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{u}}' + \tilde{\boldsymbol{\nabla}}\big(\tilde{p}' + \rho h + \tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}'_{\mathrm{R}}|^2 + \epsilon^2\tilde{\boldsymbol{u}}'_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}'_{\mathrm{D}}\big) = 0,\\ &\quad \text{where } \tilde{\zeta}' \equiv \hat{\boldsymbol{z}}\cdot\tilde{\boldsymbol{\nabla}}\times\tilde{\boldsymbol{u}}'_{\mathrm{R}}, \quad \tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}}'_{\mathrm{R}} \neq 0,\\ &\rho h_\rho + \tilde{p}'_\rho + \epsilon^2\tilde{\boldsymbol{u}}'_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}'_{\mathrm{D}\rho} = 0.\end{aligned} \tag{9.6}$$

The transformed velocity, $\tilde{\boldsymbol{u}}'_{\mathrm{R}}$, is not isopycnally divergenceless ($\tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}}'_{\mathrm{R}} \neq 0$), since the isopycnal transformation produces (cf. Eq. (8.10))

$$0 = \boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{R}} = \tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}}'_{\mathrm{R}} - \frac{1}{h_\rho}\tilde{\boldsymbol{u}}'_{\mathrm{R}\rho}\cdot\tilde{\boldsymbol{\nabla}}h, \tag{9.7}$$

and

$$\frac{1}{h_\rho}\tilde{\boldsymbol{u}}'_{\mathrm{R}\rho}\cdot\tilde{\boldsymbol{\nabla}}h = \mathrm{O}(\epsilon). \tag{9.8}$$

Consequently, the balanced equation determining the pressure $\tilde{p}'$ in the isopycnal representation of HBE resulting from directly transforming the Eulerian Helmholtz decomposition (4.2) is not expressible in a simple form in isopycnal coordinates. (Imagine transforming the order $\mathrm{O}(\epsilon^2)$ terms is Eq. (9.2) directly.) This difficulty in transforming the divergence equation to isopycnal (or isentropic) coordinates at order ϵ^2 does not occur for BE at order $\mathrm{O}(\epsilon)$, cf. [14].

9.3. *Alternative isopycnal HBE*

Because of the complicated form of the pressure equation obtained by directly transforming HBE to isopycnal coordinates, we are motivated to seek an "alternative" set of HBE which would produce a simpler balanced pressure equation in those coordinates, but would still have errors relative to the PE that are order $\mathrm{O}(\epsilon^2)$. To obtain such an alternative isopycnal HBE, we write a new Helmholtz velocity decomposition in isopycnal coordinates, as

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_{\mathrm{R}} + \epsilon\tilde{\boldsymbol{u}}_{\mathrm{D}} = \hat{z}\times\tilde{\boldsymbol{\nabla}}\tilde{\psi} + \frac{\epsilon}{h_\rho}\tilde{\boldsymbol{\nabla}}\tilde{\chi}, \tag{9.9}$$

in which $\tilde{\boldsymbol{u}}_{\mathrm{R}}$ is isopycnally divergence-free $\tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}}_{\mathrm{R}} = 0$ (primes are absent) and

$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\, h_\rho\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}_{\mathrm{D}} = 0, \tag{9.10}$$

so that $\tilde{\boldsymbol{u}}_{\mathrm{R}}$ and $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ are orthogonal with weight h_ρ. The introduction of the weight $1/h_\rho$ in $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ retains orthogonality of $\tilde{\boldsymbol{u}}_{\mathrm{R}}$ and $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ in the original (x, y, z) coordinates. We then vary the action $\mathcal{L}_{\mathrm{HBE}}$ in (9.3) at fixed (x, y, ρ, t). This action transforms into isopycnal coordinates as

$$\mathcal{L}_{\mathrm{HBE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\big[\tfrac{1}{2}\epsilon\tilde{D}|\tilde{\boldsymbol{u}} - \epsilon\tilde{\boldsymbol{u}}_{\mathrm{D}}|^2 + \tilde{D}\tilde{\boldsymbol{u}}\cdot\boldsymbol{R}(\boldsymbol{x}) - \tilde{D}\rho h - \tilde{p}(\tilde{D} - h_\rho)\big], \tag{9.11}$$

in which $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ is now given as $h_\rho^{-1}\tilde{\boldsymbol{\nabla}}\tilde{\chi}$ in Eq. (9.9). Varying $\mathcal{L}_{\mathrm{HBE}}$ at fixed (x, y, ρ, t) using the new velocity decomposition (9.9) gives

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{HBE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\big[&(\tilde{D}\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}} + \tilde{D}\boldsymbol{R}(\boldsymbol{x}))\cdot\delta\tilde{\boldsymbol{u}} - \epsilon^2\tilde{D}\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\delta\tilde{\boldsymbol{u}}_{\mathrm{D}}\\ &+ \big(\tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 + \tilde{\boldsymbol{u}}\cdot\boldsymbol{R}(\boldsymbol{x}) - \rho h - \tilde{p}\big)\delta\tilde{D} - \tilde{D}\rho\delta h + \tilde{p}\delta h_\rho - (\tilde{D} - h_\rho)\delta\tilde{p}\big].\end{aligned} \tag{9.12}$$

This is expressible in terms of variations $\delta\tilde{l}^a$, $a = 1, 2$, with respect to the Lagrange coordinate $\tilde{l}^a$ on each level surface of buoyancy, ρ, by using the definitions (8.14) again. As in the case of $\delta\mathcal{L}_{\mathrm{PE}}$ in Eq. (8.13), we substitute these definitions into the variational formula (9.12) for $\delta\mathcal{L}_{\mathrm{HBE}}$ and integrate by parts using the tangency conditions on the velocity at the boundary and requiring δh to vanish at the endpoints in ρ. In the process, we find (cf. Eqs. (8.15)–(8.17) in the PE case)

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{HBE}} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\,\Bigg\{&\tilde{D}(\tilde{D}^{-1})^i_a\delta\tilde{l}^a\left[\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_{\mathrm{R}i} + \epsilon^2\tilde{u}_{\mathrm{R}j}\tilde{u}^j_{\mathrm{D},i} + \tilde{u}^j(R_{i,j} - R_{j,i}) + (\rho h + \tilde{p})_{,i}\right]\\ &+ \epsilon^2\delta\tilde{\chi}\tilde{\boldsymbol{\nabla}}\cdot\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}} - \left(\tilde{D}\rho + \tilde{p}_\rho + \epsilon^2\left(\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}_{\mathrm{D}}\right)_\rho\right)\delta h - (\tilde{D} - h_\rho)\delta\tilde{p}\Bigg\}\end{aligned}$$

$$-\epsilon^2 \int \mathrm{d}t \oint \mathrm{d}s \int \mathrm{d}\rho \frac{\tilde{D}}{h_\rho} \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \hat{\boldsymbol{n}}|_\rho \delta\tilde{\chi}, \tag{9.13}$$

where $\hat{\boldsymbol{n}}|_\rho$ is the horizontal unit normal vector at the boundary, which lies on a level surface of ρ. Vanishing of $\delta\mathcal{L}_{\mathrm{HBE}}$ for arbitrary variations $\delta\tilde{l}^a$, δh, $\delta\tilde{p}$ and $\delta\tilde{\chi}$ within the domain of flow and on the boundary implies the following dimensionless isopycnal Hamiltonian balance equations, denoted IHBE, in a form similar to the IPE in (8.18):

$$\begin{aligned} &\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \tilde{\boldsymbol{u}}_{\mathrm{R}} - \epsilon^2 \tilde{u}_{\mathrm{D}j} \tilde{\boldsymbol{\nabla}} \tilde{u}^j_{\mathrm{R}} + f\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\nabla}}(\tilde{p} + \epsilon^2 \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{D}} + \rho h) = 0, \\ &\rho h_\rho + (\tilde{p} + \epsilon^2 \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{D}})_\rho = 0, \end{aligned} \tag{9.14}$$

where we have used the constraints

$$\begin{aligned} &\delta p: \quad \tilde{D} = h_\rho, \\ &\delta\tilde{\chi}: \quad \tilde{\boldsymbol{\nabla}} \cdot \frac{\tilde{D}}{h_\rho} \tilde{\boldsymbol{u}}_{\mathrm{R}} = 0 \text{ in the interior and } \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \hat{\boldsymbol{n}}|_\rho = 0 \text{ at the boundary.} \end{aligned} \tag{9.15}$$

These equations are augmented by the kinematic conditions

$$\partial_t h_\rho = -\tilde{\boldsymbol{\nabla}} \cdot h_\rho \tilde{\boldsymbol{u}} \quad \text{and} \quad \epsilon \tilde{w} = \frac{\mathrm{d}h}{\mathrm{d}t} = \partial_t h + \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}} h. \tag{9.16}$$

Thus, the IHBE emerge from HP, by transforming the action $\mathcal{L}_{\mathrm{HBE}}$ from Eulerian coordinates (x, y, z) to mixed Eulerian (x, y) and Lagrangian (ρ) coordinates, assuming that the Jacobian h_ρ does not vanish, and using the alternative isopycnal Helmholtz decomposition (9.9), which differs from that of (9.5) by terms in $\tilde{\boldsymbol{\nabla}} h$ of order $\mathrm{O}(\epsilon)$ and has the virtue that $\tilde{\boldsymbol{u}}_{\mathrm{R}}$ is divergenceless as a combined result of the constraints imposed by p and $\tilde{\chi}$. This process justifies choosing the Helmholtz decomposition (9.9) in which $\tilde{\boldsymbol{\nabla}}\tilde{\chi}$ in $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ acquires the weight $1/h_\rho$, so that $\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{R}} = 0$ and $\tilde{\boldsymbol{\nabla}} \cdot h_\rho \tilde{\boldsymbol{u}}_{\mathrm{D}} = \tilde{\Delta}\tilde{\chi}$, where $\tilde{\Delta}$ is the two-dimensional isopycnal Laplacian. The IHBE model is balanced, since the time derivative $\tilde{\boldsymbol{u}}_{\mathrm{D}t}$ is absent in the elliptic equation that determines the IHBE pressure

$$\tilde{\Delta}(\tilde{p} + \rho h) + \tilde{\boldsymbol{\nabla}} \cdot (f\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{u}} + \epsilon(\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}})\tilde{\boldsymbol{u}}_{\mathrm{R}} + \epsilon^2 \tilde{u}_{\mathrm{R}j} \tilde{\boldsymbol{\nabla}} \tilde{u}^j_{\mathrm{D}}) = 0. \tag{9.17}$$

This balanced pressure equation (obtained from the horizontal *isopycnal* divergence of the motion equation in (9.14)) is to be solved with Neumann boundary conditions

$$\hat{\boldsymbol{n}}|_\rho \cdot (\tilde{\boldsymbol{\nabla}}(\tilde{p} + \rho h) + f\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{u}} + \epsilon^2 \tilde{u}_{\mathrm{R}j} \tilde{\boldsymbol{\nabla}} \tilde{u}^j_{\mathrm{D}}) + \epsilon\kappa |\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 = 0, \tag{9.18}$$

where κ is the curvature at the boundary and $\hat{\boldsymbol{n}}|_\rho$ is its horizontal unit normal vector defined on a level suface of ρ. Finally, the relation between $\tilde{\chi}$ and w in IHBE is obtained from the kinematic condition (9.16) as

$$\epsilon \tilde{w}_\rho - (\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}} h)_\rho = -\tilde{\boldsymbol{\nabla}} \cdot (h_\rho \tilde{\boldsymbol{u}}) = -\hat{\boldsymbol{z}} \cdot \tilde{\boldsymbol{\nabla}}\tilde{\psi} \times \tilde{\boldsymbol{\nabla}} h_\rho - \epsilon\tilde{\Delta}\tilde{\chi}. \tag{9.19}$$

The last equation is still properly ordered in ϵ, since $|\tilde{\boldsymbol{\nabla}} h| = \mathrm{O}(\epsilon)$, as discussed earlier, cf. Eq. (8.9).

10. Hamiltonian formulation of isopycnal HBE

10.1. Legendre transformation

In order to Legendre transform the action $\mathcal{L}_{\mathrm{HBE}}$ in (9.11) we first compute the momentum $\tilde{\pi}_a$, with $a = 1, 2$, which is canonically conjugate to l^a at order $\mathrm{O}(\epsilon)$. By definition,

$$\tilde{\pi}_a=\frac{\delta\mathcal{L}_{\mathrm{HBE}}}{\delta\tilde{l}^a_{,t}}=\frac{\delta\tilde{u}^i}{\delta\tilde{l}^a_{,t}}\frac{\delta\mathcal{L}_{\mathrm{HBE}}}{\delta\tilde{u}^i}$$
$$=-(\tilde{D}^{-1})^i_a\frac{\delta\mathcal{L}_{\mathrm{HBE}}}{\delta\tilde{u}^i}$$
$$=-(\tilde{D}^{-1})^i_a\tilde{D}(\epsilon\tilde{u}_{\mathrm{R}i}+R_i), \tag{10.1}$$

where we have used Eq. (9.13). Thus, the momentum density defined by $\tilde{\boldsymbol{m}}=-\tilde{\pi}_a\tilde{\boldsymbol{\nabla}}\tilde{l}^a$ is given by

$$\tilde{\boldsymbol{m}}=\tilde{D}(\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}}+\boldsymbol{R}). \tag{10.2}$$

Consequently, we have the usual relation

$$\tilde{\pi}_a\tilde{l}^a_{,t}=\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{u}}. \tag{10.3}$$

The Legendre transformation to the Hamiltonian for IHBE is then expressed as

$$H_{\mathrm{IBE}}=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho(\tilde{\pi}_a\tilde{l}^a_{,t}-L_{\mathrm{HBE}})$$
$$=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho[\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{u}}-\tilde{D}\tilde{\boldsymbol{u}}\cdot\boldsymbol{R}-\tfrac{1}{2}\epsilon\tilde{D}|\tilde{\boldsymbol{u}}-\epsilon\tilde{\boldsymbol{u}}_{\mathrm{D}}|^2+\tilde{D}\rho h+\tilde{p}(\tilde{D}-h_\rho)]$$
$$=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\left[\frac{1}{2\epsilon\tilde{D}}|\tilde{\boldsymbol{m}}-\mathcal{D}\boldsymbol{R}|^2+\tilde{D}\rho h+\tilde{p}(\tilde{D}-h_\rho)+\epsilon(\tilde{\boldsymbol{m}}-\tilde{D}\boldsymbol{R})\cdot\tilde{\boldsymbol{u}}_{\mathrm{D}}\right], \tag{10.4}$$

which is a sum of kinetic and potential energies, plus constraints on H_{IBE}. Again this is a Routhian; we are not Legendre transforming the Lagrange multipliers, $\tilde{p}$, h and $\tilde{\chi}$, since these quantities have no conjugate momenta. The variational derivatives of $H_{\mathrm{IBE}}(\tilde{\boldsymbol{m}},\tilde{D})$ are given by

$$\delta H_{\mathrm{IBE}}=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\left\{\tilde{\boldsymbol{u}}\cdot\delta\tilde{\boldsymbol{m}}+\left(-\tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2-\tilde{\boldsymbol{u}}\cdot\boldsymbol{R}+\tilde{p}+\rho h\right)\delta\tilde{D}\right.$$
$$\left.+\left(\tilde{D}\rho+\left(\tilde{p}+\epsilon^2\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}_{\mathrm{D}}\right)_\rho\right)\delta h+(\tilde{D}-h_\rho)\delta\tilde{p}+\delta\tilde{\chi}\tilde{\boldsymbol{\nabla}}\cdot\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}}\right\}$$
$$-\int \mathrm{d}\rho\oint \mathrm{d}s\,\delta\tilde{\chi}\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\hat{\boldsymbol{n}}|_\rho, \tag{10.5}$$

where we have dropped a boundary term arising from integrating by parts in ρ. Thus, the quantities $\tilde{p}$, h and $\tilde{\chi}$ are three Lagrange multipliers that impose the following constraints:

$$\begin{aligned}
&\delta\tilde{p}:\quad \tilde{D}-h_\rho=0,\quad \text{volume preservation of the flow;}\\
&\delta h:\quad \tilde{D}\rho+\left(\tilde{p}+\epsilon^2\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}_{\mathrm{D}}\right)_\rho=0,\quad \text{modified hydrostasy;}\\
&\delta\tilde{\chi}:\quad \tilde{\boldsymbol{\nabla}}\cdot\frac{\tilde{D}}{h_\rho}\tilde{\boldsymbol{u}}_{\mathrm{R}}=0 \text{ in the interior and } \tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\hat{\boldsymbol{n}}|_\rho=0 \text{ at the boundary.}
\end{aligned} \tag{10.6}$$

10.2. Lie–Poisson bracket in isopycnal variables and Kelvin's theorem for IHBE

The change of variables from $\tilde{\pi}_a$ and $\tilde{l}^a$ to $\tilde{\boldsymbol{m}}$ and $\tilde{D}$ induces the following Lie–Poisson bracket from the canonical Poisson bracket in $\tilde{\pi}_a$ and $\tilde{l}^a$ (cf. Eq. (7.13)):

$$\{F, G\}\left(\tilde{\boldsymbol{m}}, \tilde{D}\right) = -\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho \left[\frac{\delta F}{\delta \tilde{m}_i}\left((\partial_j \tilde{m}_i + \tilde{m}_j \partial_i)\frac{\delta G}{\delta \tilde{m}_j} + \tilde{D}\partial_i \frac{\delta G}{\delta \tilde{D}}\right) + \frac{\delta F}{\delta \tilde{D}}\partial_j \tilde{D}\frac{\delta G}{\delta \tilde{m}_j}\right], \tag{10.7}$$

where $\partial_j = \partial/\partial x^j$, $j = 1, 2$, operates on all terms it multiplies to its right. The corresponding equations of motion are given in Lie–Poisson Hamiltonian form by

$$\begin{aligned}\partial_t \tilde{m}_i &= \{\tilde{m}_i, H_{\mathrm{IBE}}\} = -(\partial_j \tilde{m}_i + \tilde{m}_j \partial_i)\tilde{u}^j - \tilde{D}\partial_i\left(-\tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 - \tilde{\boldsymbol{u}}\cdot\boldsymbol{R} + \tilde{p} + \rho h\right),\\ \partial_t \tilde{D} &= \{\tilde{D}, H_{\mathrm{IBE}}\} = -\partial_j \tilde{D}\tilde{u}^j.\end{aligned} \tag{10.8}$$

These are the IHBE (9.14) and (9.16) now expressed in terms of $\tilde{\boldsymbol{m}}$ and $\tilde{D}$. Substituting into (10.8) the definitions of $\tilde{\boldsymbol{m}}$ given in Eq. (10.2) recovers the IHBE motion equation in Kelvin theorem form, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}} + \boldsymbol{R}) + (\epsilon\tilde{u}_{\mathrm{R}j} + R_j)\tilde{\boldsymbol{\nabla}}\tilde{u}^j = \tilde{\boldsymbol{\nabla}}\left(-\tilde{p} - \rho h + \tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 + \tilde{\boldsymbol{u}}\cdot\boldsymbol{R}\right). \tag{10.9}$$

Hence, we obtain Kelvin's theorem for IHBE:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)}(\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}} + \boldsymbol{R})\cdot\mathrm{d}\boldsymbol{x} &= \oint_{\gamma(t)}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}} + \boldsymbol{R}) + (\epsilon\tilde{u}_{\mathrm{R}j} + R_j)\tilde{\boldsymbol{\nabla}}\tilde{u}^j\right]\cdot\mathrm{d}\boldsymbol{x}\\ &= \oint_{\gamma(t)}\tilde{\boldsymbol{\nabla}}\left(-\tilde{p} - \rho h + \tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 + \tilde{\boldsymbol{u}}\cdot\boldsymbol{R}\right)\cdot\mathrm{d}\boldsymbol{x} = 0.\end{aligned} \tag{10.10}$$

Application of Stokes theorem implies, as before, that the flux of total vorticity through an isopycnal surface is invariant. Rearranging the IHBE motion equation (10.9) using the vector identity (3.4) gives

$$\epsilon\partial_t\tilde{\boldsymbol{u}}_{\mathrm{R}} - \tilde{\boldsymbol{u}}\times(\tilde{\boldsymbol{\nabla}}\times(\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}} + \boldsymbol{R})) + \tilde{\boldsymbol{\nabla}}\left(\tilde{p} + \rho h + \tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 + \epsilon^2\tilde{\boldsymbol{u}}_{\mathrm{R}}\cdot\tilde{\boldsymbol{u}}_{\mathrm{D}}\right) = 0, \tag{10.11}$$

whose isopycnal curl in combination with the continuity equation for $\tilde{D} = h_\rho$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Q} = 0 \quad \text{with } \tilde{Q} = \frac{1}{h_\rho}\hat{\boldsymbol{z}}\cdot\tilde{\boldsymbol{\nabla}}\times(\epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}} + \boldsymbol{R}) = \frac{1}{\tilde{D}}\hat{\boldsymbol{z}}\cdot\tilde{\boldsymbol{\nabla}}\times\tilde{\boldsymbol{m}}/\tilde{D}. \tag{10.12}$$

This is advection of the IHBE potential vorticity, which may also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f + \epsilon\tilde{\omega}_{\mathrm{R}}}{h_\rho}\right) = 0, \tag{10.13}$$

where $\tilde{\omega}_{\mathrm{R}} \equiv \hat{\boldsymbol{z}}\cdot\tilde{\boldsymbol{\nabla}}\times\tilde{\boldsymbol{u}}_{\mathrm{R}} = \tilde{\Delta}\tilde{\psi}$ is the scalar isopycnal vorticity. Hence, the IHBE conserve both the energy,

$$E = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\,\tilde{D}\left(\tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{R}}|^2 + \rho h\right) \quad \text{with } \tilde{D} = h_\rho, \tag{10.14}$$

as well as the Casimirs,

$$C_\Phi = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho\,\tilde{D}\Phi(\tilde{Q}), \tag{10.15}$$

for any function Φ. The Casimirs (10.15) also Poisson-commute under the Lie–Poisson bracket (10.7) with any functional of $\tilde{\boldsymbol{m}}$ and $\tilde{D}$, i.e.,

$$\{C_\Phi, G\} = 0 \quad \text{for all } G(\tilde{\boldsymbol{m}}, \tilde{D}). \tag{10.16}$$

Thus, the weighted Helmholtz decomposition (9.5) has led via HP to the system IHBE in (9.14), in which all of the structure of the Eulerian HBE is preserved and the hydrostatic condition is slightly altered in isopycnal variables.

Remark (on single layer IHBE for shallow water). We specialize IHBE to a single fluid layer of thickness, η. For this, we decompose the horizontal velocity as

$$u = u_{\mathrm{R}} + \epsilon u_{\mathrm{D}} = \hat{z} \times \nabla\psi + \frac{\epsilon}{\eta}\nabla\chi. \tag{10.17}$$

Substituting this velocity decomposition into the correspondingly modified action, cf. Eq. (9.11),

$$\mathcal{L}_{SWBE} = \int \mathrm{d}t \int \mathrm{d}x\,\mathrm{d}y \left[\frac{\epsilon}{2}\eta|u - \epsilon u_{\mathrm{D}}|^2 + \eta u \cdot R(x) - \frac{(\eta - b)^2}{2\epsilon\mathcal{F}}\right], \tag{10.18}$$

yields Hamiltonian balance equations for rotating shallow water dynamics in essentially the same form as IHBE. Namely, cf. Eq. (9.14),

$$\begin{aligned} &\epsilon\frac{\mathrm{d}}{\mathrm{d}t}u_{\mathrm{R}} - \epsilon^2 u_{\mathrm{D}j}\nabla u_{\mathrm{R}}^j + f\hat{z} \times u + \nabla h = 0 \quad \text{with } h = \frac{\eta - b}{\epsilon\mathcal{F}}, \\ &\partial_t\eta + \nabla \cdot \eta u = 0. \end{aligned} \tag{10.19}$$

In these equations $f = \hat{z} \cdot \nabla \times R$, while h is the nondimensional height of the free surface above the equilibrium level, $z = 0$. The bottom is fixed at $z = -b(x, y)$ and $\mathcal{F}$ denotes the squared ratio of the typical horizontal scale of motion to the Rossby deformation radius.

The motion equations (10.19) for shallow water HBE coincide with those for the shallow water BEM model discussed in [4,5]. However, because of the weight η in the Helmholtz decomposition (10.17) for u, the equation of balance for χ is different. Instead, the decomposition (10.17) and the divergence of eq. (10.19) combine to give

$$\nabla \cdot (f\nabla\psi) - \Delta h = \epsilon[q, \chi]. \tag{10.20}$$

Here q is the potential vorticity, which is advected, cf. Eqs. (10.12) and (10.13),

$$q = \frac{f + \epsilon\Delta\psi}{\eta}, \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + u \cdot \nabla q = 0. \tag{10.21}$$

These shallow-water HBE conserve the energy,

$$E_{\mathrm{SWBE}} = \int \mathrm{d}x\,\mathrm{d}y\left(\tfrac{1}{2}\epsilon\eta|u_{\mathrm{R}}|^2 + \tfrac{1}{2}h^2\right), \tag{10.22}$$

as well as the usual Casimirs for the corresponding Lie–Poisson bracket in the Hamiltonian formulation,

$$C_\Phi = \int \mathrm{d}x\,\mathrm{d}y\,\eta\Phi(q), \tag{10.23}$$

for any function Φ. We will discuss the Hamiltonian formulation of the shallow-water HBE and its implications in more detail elsewhere.

11. Summary of IHBE in comparison to IPE and other Isopycnal BE in the literature

The two families of Hamiltonian equations HBE and IHBE are only order $O(\epsilon^2)$ different from PE and IPE, respectively. In addition, HBE and IHBE are balanced and retain an exact Kelvin theorem (which implies exact potential vorticity advection). Our objective has been to create such equations by making approximations in HP for the ideal fluid equations. We have also explored commonalities among the Lie–Poisson Hamiltonian descriptions, conservation laws and Kelvin circulation properties of the resulting models. Section 5 compares the HBE with PE and with other BEs in the literature. Here we summarize and compare the isopycnal models IPE and IHBE. We also discuss the differences between IHBE and another Isopycnal BE in the literature [14].

The motion equation and hydrostatic balance for the IPE model are recalled from Eq. (8.18):

$$\epsilon \frac{d}{dt}\tilde{\boldsymbol{u}} + f\hat{z} \times \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\nabla}}(\tilde{p} + \rho h) = 0, \qquad \rho h_\rho + \tilde{p}_\rho = 0. \tag{11.1}$$

In IHBE, we first write the Helmholtz velocity decomposition in isopycnal coordinates, as in Eq. (9.9) with an additional weight in the divergent component,

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_{\mathrm{R}} + \epsilon\tilde{\boldsymbol{u}}_{\mathrm{D}} = \hat{z} \times \tilde{\boldsymbol{\nabla}}\tilde{\psi} + \frac{\epsilon}{h_\rho}\tilde{\boldsymbol{\nabla}}\tilde{\chi}. \tag{11.2}$$

The dimensionless IHBE then appear in a similar form to Eqs. (11.1) for the IPE,

$$\begin{aligned} &\epsilon \frac{d}{dt}\tilde{\boldsymbol{u}}_{\mathrm{R}} + \epsilon^2 \tilde{u}_{\mathrm{R}j}\tilde{\boldsymbol{\nabla}}\tilde{u}_{\mathrm{D}}^j + f\hat{z} \times \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\nabla}}(\tilde{p} + \rho h) = 0, \\ &\rho h_\rho + (\tilde{p} + \epsilon^2 \tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{D}})_\rho = 0. \end{aligned} \tag{11.3}$$

Both models impose $\tilde{D} = h_\rho$ and $\tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \hat{\boldsymbol{n}}|_\rho = 0$ at the boundary. Both systems are closed by applying the kinematic conditions,

$$\begin{aligned} &\epsilon\tilde{w} = \frac{dh}{dt} = \partial_t h + \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}} h \quad &&\text{(the definition of vertical velocity)}, \\ &\partial_t h_\rho = -\tilde{\boldsymbol{\nabla}} \cdot h_\rho \tilde{\boldsymbol{u}} &&\text{(incompressibility)}, \end{aligned} \tag{11.4}$$

in which $|\partial_t h|$ and $|\tilde{\boldsymbol{\nabla}} h|$ are order $O(\epsilon)$.

We now write the IPE motion equation as "IHBE with order $O(\epsilon^2)$ perturbations", cf. Eq. (10.11),

$$\begin{aligned} &\epsilon\partial_t\tilde{\boldsymbol{u}}_{\mathrm{R}} - \tilde{\boldsymbol{u}} \times (f + \epsilon\tilde{\omega}_{\mathrm{R}})\hat{z} + \tilde{\boldsymbol{\nabla}}\big(\tilde{p} + \rho h + \tfrac{1}{2}\epsilon|\tilde{\boldsymbol{u}}_{\mathrm{R}}|^2 + \epsilon^2\tilde{\boldsymbol{u}}_{\mathrm{R}} \cdot \tilde{\boldsymbol{u}}_{\mathrm{D}}\big) \\ &\quad = \epsilon^2\big(- \partial_t\tilde{\boldsymbol{u}}_{\mathrm{D}} + \tilde{\boldsymbol{u}} \times (\tilde{\boldsymbol{\nabla}} \times \tilde{\boldsymbol{u}}_{\mathrm{D}}) - \tfrac{1}{2}\epsilon\tilde{\boldsymbol{\nabla}}|\tilde{\boldsymbol{u}}_{\mathrm{D}}|^2\big), \end{aligned} \tag{11.5}$$

where $\tilde{\omega}_{\mathrm{R}} = \tilde{\boldsymbol{\nabla}} \times \tilde{\boldsymbol{u}}_{\mathrm{R}}$ and the left-hand side vanishes for IHBE. Thus, IPE can be regarded as IHBE driven by an order $O(\epsilon^2)$ "Lorentz force", $\boldsymbol{E} + \tilde{\boldsymbol{u}} \times \boldsymbol{B}$, in which $\boldsymbol{E} = -\partial_t\boldsymbol{A} - \tilde{\boldsymbol{\nabla}}\phi$ and $\boldsymbol{B} = \tilde{\boldsymbol{\nabla}} \times \boldsymbol{A}$. The vector and scalar potentials are identified as $\boldsymbol{A} = \tilde{\boldsymbol{u}}_{\mathrm{D}}$ and $\phi = \epsilon|\tilde{\boldsymbol{u}}_{\mathrm{D}}|^2/2$.

The same comparison can be made between IHBE and IPE in Kelvin theorem form, cf. Eqs. (8.20) and (10.10). Namely,

$$\underbrace{\frac{d}{dt}\oint_{\tilde{\gamma}(t)} (\boldsymbol{R} + \epsilon\tilde{\boldsymbol{u}}) \cdot d\boldsymbol{x}}_{\text{IPE}=0} = \underbrace{\frac{d}{dt}\oint_{\tilde{\gamma}(t)} (\boldsymbol{R} + \epsilon\tilde{\boldsymbol{u}}_{\mathrm{R}}) \cdot d\boldsymbol{x}}_{\text{IHBE}=0} + \underbrace{\frac{d}{dt}\oint_{\tilde{\gamma}(t)} (\epsilon^2\tilde{\boldsymbol{u}}_{\mathrm{D}}) \cdot d\boldsymbol{x}}_{\text{Balance}}, \tag{11.6}$$

where the contour $\tilde{\gamma}(t)$ moves with velocity $\tilde{\boldsymbol{u}}$ on an isopycnal surface. The last circulation integral in this case is nonvanishing,

$$\oint_{\tilde{\gamma}(t)} \epsilon^2\tilde{\boldsymbol{u}}_{\mathrm{D}} \cdot d\boldsymbol{x} = \oint_{\tilde{\gamma}(t)} \frac{\epsilon^2}{h_\rho} d\tilde{\chi} \neq 0. \tag{11.7}$$

The order $O(\epsilon^2)$ nonzero circulation of $\tilde{\boldsymbol{u}}_{\mathrm{D}}$ is the sense in which IHBE differs from both IPE and the Isopycnal BE in [14], which has the same circulation theorem as IPE. The latter model also has exact hydrostasy but no conserved energy. Like HBE, the IHBE model has two degrees of freedom, which can be taken as $\tilde{\psi}$ and h_ρ.

11.1. Geometrical interpretation of Kelvin's theorem

As explained after equation (2.13), Kelvin's theorem in the canonical Hamiltonian formulation is the statement of invariance of the action integral $\oint p\, \mathrm{d}q$ on level surfaces of buoyancy in either Eulerian or Lagrangian variables. From this viewpoint, the decomposition of the IPE circulation integral on an isopycnal surface in (11.6) may also be understood geometrically, as follows. We shall regard $\Omega = (\boldsymbol{R} + \epsilon\tilde{\boldsymbol{u}}_\mathrm{R}) \cdot \mathrm{d}\boldsymbol{x}$ (the IHBE circulation 1-form) as a connection on a fiber bundle. Relative to this connection, horizontal vector fields X_h satisfy $X_\mathrm{h} \cdot \Omega = 0$. Consequently, the IPE circulation integral in (11.6) divides into horizontal and vertical components relative to this connection, as

$$I_{\mathrm{IPE}} = \underbrace{\oint_{\tilde{\gamma}(t)} (\boldsymbol{R} + \epsilon\tilde{\boldsymbol{u}}) \cdot \mathrm{d}\boldsymbol{x}}_{\text{Total}} = \underbrace{\oint_{\tilde{\gamma}(t)} (\boldsymbol{R} + \epsilon\tilde{\boldsymbol{u}}_\mathrm{R}) \cdot \mathrm{d}\boldsymbol{x}}_{\text{Horizontal}} + \underbrace{\oint_{\tilde{\gamma}(t)} (\epsilon^2\tilde{\boldsymbol{u}}_\mathrm{D}) \cdot \mathrm{d}\boldsymbol{x}}_{\text{Vertical}}. \tag{11.8}$$

Applying Stokes theorem to these circulation integrals gives

$$\begin{aligned} I_{\mathrm{IPE}} &= \int\!\!\int_{\tilde{S}(t)} \left(f + \epsilon\tilde{\Delta}\tilde{\psi}\right) \mathrm{d}x \wedge \mathrm{d}y + \int\!\!\int_{\tilde{S}(t)} \left[\frac{\epsilon^2}{h_\rho}, \tilde{\chi}\right] \mathrm{d}x \wedge \mathrm{d}y \\ &= \int\!\!\int_{\tilde{S}(t)} \left(\frac{f + \epsilon\tilde{\Delta}\tilde{\psi}}{h_\rho}\right) \mathrm{d}\tilde{l}^1 \wedge \mathrm{d}\tilde{l}^2 + \int\!\!\int_{\tilde{S}(t)} \frac{1}{h_\rho}\left[\frac{\epsilon^2}{h_\rho}, \tilde{\chi}\right] \mathrm{d}\tilde{l}^1 \wedge \mathrm{d}\tilde{l}^2, \end{aligned} \tag{11.9}$$

where the Lagrangian surface element satisfies $\tilde{l}^1 \wedge \mathrm{d}\tilde{l}^2 = h_\rho\, \mathrm{d}x \wedge \mathrm{d}y$. Also, $\tilde{\Delta}$ denotes the Laplacian and square brackets denote the Jacobian expressed in isopycnal horizontal coordinates. Thus, the IPE circulation conserved around any contour $\tilde{\gamma}(t)$ moving with the fluid on an isopycnal surface is the area enclosed by that contour, weighted by its vorticity in Eulerian coordinates and by its potential vorticity in Lagrangian coordinates. It consists of the contribution conserved by IHBE dynamics plus the contribution arising from the divergent part of the IPE horizontal velocity.

The geometrical significance of the decomposition (11.8) emerges from the relation of the Kelvin circulation integral to the Poincaré action integral in the canonical Hamiltonian formulation. In the canonical Hamiltonian formulations of IPE and IHBE, the corresponding Kelvin circulation integrals are expressed as, cf. Eqs. (2.13) and (10.1),

$$\begin{aligned} I_{\mathrm{IPE}} &= -\oint_{\tilde{\gamma}(t)} (\tilde{\pi}_a^{\mathrm{IPE}}/h_\rho)\, \mathrm{d}\tilde{l}^a = -\oint_{\tilde{\gamma}(t)} (\tilde{\pi}_a^{\mathrm{IHBE}}/h_\rho)\, \mathrm{d}\tilde{l}^a + \oint_{\tilde{\gamma}(t)} \frac{\epsilon^2}{h_\rho}\, \mathrm{d}\tilde{\chi} \\ &= -\int\!\!\int_{\tilde{S}(t)} \mathrm{d}(\tilde{\pi}_a^{\mathrm{IPE}}/h_\rho) \wedge \mathrm{d}\tilde{l}^a \\ &= -\int\!\!\int_{\tilde{S}(t)} \mathrm{d}(\tilde{\pi}_a^{\mathrm{IHBE}}/h_\rho) \wedge \mathrm{d}\tilde{l}^a + \int\!\!\int_{\tilde{S}(t)} \mathrm{d}\left(\frac{\epsilon^2}{h_\rho}\right) \wedge \mathrm{d}\tilde{\chi}, \end{aligned} \tag{11.10}$$

where $\tilde{\pi}_a^{\mathrm{IPE}}$ and $\tilde{\pi}_a^{\mathrm{IHBE}}$ denote the momentum densities canonically conjugate to the isopycnal Lagrangian coordinate fields $\tilde{l}^a$, $a = 1, 2$, for the IPE and IHBE flows, respectively. Thus, the invariant IPE circulation integral I_{IPE} around any isopycnal contour $\tilde{\gamma}(t)$ moving with the flow may be interpreted geometrically as the sum of the areas enclosed by the horizontal and vertical lifts of that contour onto Poincaré surfaces embedded in the phase spaces of the corresponding components of the IPE system. The horizontal and vertical lifts of the contour are determined relative to the connection defined by the IHBE circulation 1-form.

11.2. Concluding remark

Kelvin's circulation theorem is a consequence of HP for ideal fluids and corresponds to invariance of the Poincaré action integral $\oint p\, \mathrm{d}q$ in classical Hamiltonian particle dynamics. Asymptotic expansions of HP for fluids preserve the Kelvin circulation property and lead systematically to the balanced HBE and IHBE models. The IHBE model preserves the same balance and Kelvin circulation properties as HBE and is expressed in isopycnal variables. The HBE and IHBE models each have two degrees of freedom, as opposed to the three degrees of freedom for PE and IPE. Up to a redefinition of pressure the HBE approximation is equivalent to the BEM model, which performs satisfactorily in comparisons with EB (and with the equations of rotating shallow water dynamics in the case of constant density) [3]. Both HBE and IHBE have Lie–Poisson Hamiltonian formulations and conserve energy and potential vorticity on parcels exactly. Also, their steady flows are relative equilibria, i.e., critical points of a sum of conserved quantities including the constrained Hamiltonian.

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